Lectures on Directed Graphs, Shandong University summer school on Graph Theory July 6-10, 2020

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July 5, 2020

Below BJG refers to the onlie version of Bang-Jensen and Gutin: Digraphs first edition available at http://www.cs.rhul.ac.uk/books/dbook/main.pdf

1 July 6, 2020

- 14:00–15:30: Network flows and their applications to (di)graph problems Examples of topics to be covered:

  (a) Flow decomposition

  (b) Integrality theorem

  (c) Max-Flow-Min-Cut Theorem

  (d) Hoffman's circulation theorem

  (e) Minimum value flows

  (f) Unit capacity networks

  (g) Path-Cycle covering number in polynomial time

  (h) Mengers theorem.

  Material BJG Chapter 3 and Section 7.3.

- 16:00–17:30: Various topics on digraphs. Examples are

  (a) Hamiltonian paths and cycles. BJG Chapter 5.

  (b) Arc-disjoint branchings (Edmonds' branching theorem). BJG 9.5.

  (c) (arc)-disjoint paths with prescribed ends (complexity and acyclic case). BJG 9.2

  (d) Spanning eulerian trails and Eulerian factors in digraphs. From Papers.

  (e) Covering the vertices of a digraph by disjoint paths and cycles. BJG 5.2

(f) Submodularity of degree functions of digraphs. BJG 7.1 and part of 7.3
(g) Strong orientations of graphs. BJG 1.6

- 19:00–20:30: Introduction to some Classes of digraphs. This includes
  (a) tournaments and semicomplete digraphs,
  (b) Locally semicomplete digraphs
    – In-semicomplete digraphs and Out-semicomplete digraphs.
  (c) Extended semicomplete digraphs
  (d) Quasi-transitive digraphs.
  (e) Path-mergeable digraphs.

  Material mostly from BJG 4.8-4.11.

2 July 7, 2020

- 14:00–15:30:
  (a) Structure of locally semicomplete digraphs. BJG 4.10-4.11
  (b) Structure of quasi-transitive digraphs. BJG 4.8
  (c) Longest cycles in extended semicomplete digraphs and semicomplete bipartite digraphs. Material BJG 5.7-5.8.
- 16:00–17:30:
  (a) Examples of how to apply the structural characterizations for locally semicomplete and quasi-transitive digraphs. In particular for quasi-transitive digraphs (path covering number, hamiltonian cycles). That part is BJG 5.9
  (b) Mader's splitting theorem and Frank's algorithm for increasing the arc-connectivity of a digraph optimally. BJG Section 7.5-7.6.
- 19:00–20:30: Exercises:
  BJG: 3.28(a)+(b), 3.33, 3.34, 3,35, 3.45, 3.55, 3.56 (Hint: use the integrality theorem) 3.59,3.65, 3.67, 3.70, 5.8

3 July 8, 2020

- 14:00–15:30: Orientations of graphs and submodular flows. Examples:
  (a) recognizing underlying graphs of locally semicomplete digraphs and quasi-transitive graphs
  (b) Gallai-Roy-Vitaver Theorem.

(c) Nash-Williams orientation theorem
(d) Submodular flows
(e) Arc reversals

Material BJG Chapter 8.

- 16:00–17:30: Disjoint directed and undirected subdigraphs in digraphs. Based on papers by Bang-Jensen and Kriesell. PDFs will be made available
- 19:00–20:30: Exercises
  - BJG 4.18, 4.30
  - BJG 5.13, 5.14
  - BJG 7.15, 9.34 (hint: look at the proof of Lemma 7.6.2).
  - 7.11, 7.20, 7.26,7.27, 7.28, 7.30, 7.36, 7.38, 7.47, 7.50

4 July 9, 2020

- 14:00–15:30: More results on locally semicomplete digraphs and quasi-transitive digraphs. Examples are linking problems.
- 16:00–17:30: Partition problems for digraphs. Based on research papers. PDFs will be made available
- 19:00–20:30: Exercises
  - BJG 4.20, 4.31, 4.33 (hint: use Lemma 4.13 and Lemma 4.14), 4.35
  - BJG 8.1, 8.7, 8.9, 8.20, 8.39, 8.46, 8.47, 8.48 (hint use the approach in Section 8.7.1 with D as the reference orientation and use flow decomposition on the associated flow which shows how to obtain D' from D by arc reversals), 8.65
  - BJG 9.59

5 July 10, 2020

- 14:00–15:30: Digraphs contra edge-coloured graphs. Material BJG section 11.1. Among many other things, we will show how results on cycles in bipartite digraphs are closely connected to results an cycles in 2-edge-coloured bipartite graphs.
- 16:00–17:30: Antistrong digraphs and good acyclic orientations of graphs. Based on recent works. The papers are available on ArXiv.

- 19:00–20:30: Exercises:
  - BJG 7.48
  - BJG 9.1, 9.7, 9.26, 9.27
  - This exercise is about weak linkings in (almost) cyclic digraphs.

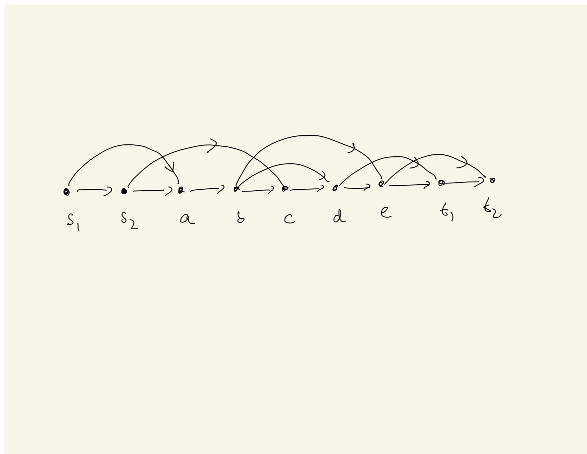

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Figure 1: An ayclic digraph

Let D be the acyclic digraph in Figure 1. Which path in D' corresponds to the solution $P_1 = s_1{\to}a{\to}b{\to}d{\to}t_1$ and $P_2 = s_2{\to}c{\to}e{\to}t_2$?

* Which solution in D corresponds to the following path in D'?

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \to \begin{pmatrix} a \\ s_2 \end{pmatrix} \to \begin{pmatrix} a \\ c \end{pmatrix} \to \begin{pmatrix} b \\ c \end{pmatrix} \to \begin{pmatrix} e \\ c \end{pmatrix} \to \begin{pmatrix} e \\ d \end{pmatrix} \to \begin{pmatrix} e \\ t_2 \end{pmatrix} \to \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

* A **feedback vertex set** of a digraph $D = (V, A)$ is a set of vertices $X \subseteq V$ such that the digraph $D' = D - X$ obtained by deleting all vertices in X is acyclic. So for acyclic digraphs the size of a minimum feedback vertex set is zero.

  Prove that the 2-path problem can be solved in polynomial time on digraphs that have a feedback vertex set of size one. Hint: use that the k-path problem is polynomial for acyclic digraphs when k is a constant.

* Generalize your solution above to digraphs with a feedback vertex set of size at most 2.

– Let $D = (V, A)$ be a digraph in which $d^+_D(v) + d^-_D(v)$ is even for every vertex $v \in V$.

  (a) Argue that it is possible to reverse some arcs in D so that the resulting digraph D' has

  $$d^+_{D'}(v) = d^-_{D'}(v) \text{ for all } v \in V \tag{1}$$

  (b) Explain how we can use flows to find a set of arcs $\tilde{A} \subset A$ to reverse such that the resulting digraph D' satisfies (1).

  (c) Extend your solution above so that you can find the minimum number of arcs we need to reverse to get a digraph satisfying (1).

  (d) Can you akso find the maximum number of arcs we need to reverse to get a digraph satisfying (1)?

– More exercises may be listed later, perhaps after the course has started.

6 July 11, 2020

Exam from 14:00 to 16:00 Exercises to be announced later.