Homework 1

(due Friday, March 10, 2023)

1. Prove that for any $n > 1$,

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is not an integer.

2. Show that there are infinitely many prime numbers of the form $4m+3$ with $m \in \mathbb{N}$.

3. Prove the following slightly stronger version of Dirichlet's approximation theorem: Let $Q \geq 1$ be a positive real number. Then for any real number α, there exist integers a, q with $1 \leq q \leq Q$ and $(a, q) = 1$, such that

$$\left| \alpha - \frac{a}{q} \right| < \frac{1}{qQ}.$$

4. Let α and β be positive real numbers. Let

$$A = \{[\alpha n] : n \in \mathbb{N}^*\}, \quad B = \{[\beta n] : n \in \mathbb{N}^*\}.$$

Prove that A and B take all positive integers without repetition (i.e. $A \cup B = \mathbb{N}^*$ and $A \cap B = \emptyset$) if and only if α and β are irrational and

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$