Homework 2

(due Friday, March 17, 2023)

1. Show that
$$\sum_{d|n} \mu^2(d) = 2^{\omega(n)}.$$

2. A perfect number is a positive integer that is equal to the sum of its proper divisors. That is, n is a perfect number if and only if
$$\sigma(n) = 2n,$$
where $\sigma(n)$ is the sum of divisors of n. Prove that n is a perfect number if and only if
$$\sum_{d|n} \frac{1}{d} = 2.$$

3. Let $\tau(n)$ denote the number of divisors of n. Prove that for $\operatorname{Re} s > 1$,
$$\sum_{n=1}^{\infty} \frac{\tau^2(n)}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)}.$$

4. Let q, n be positive integers. Define the *Ramanujan sum* $c_q(n)$ by
$$c_q(n) = \sum_{\substack{1 \le h \le q \\ (h,q)=1}} e\left(\frac{nh}{q}\right)$$
where $e(u) := e^{2\pi i u}, u \in \mathbb{R}$.

   1). Show that
$$c_q(n) = \sum_{d|(q,n)} d\mu\left(\frac{q}{d}\right).$$

2). Let $t = q/(n, q)$. Show that

$$c_q(n) = \frac{\mu(t)\varphi(q)}{\varphi(t)}.$$