Chapter 4

The distribution of prime numbers: elementary method

Recall the prime counting function $\pi(x)$ is defined to be the number of prime numbers not exceeding x. To study the distribution of prime numbers, we introduce the following functions:

$$\psi(x) = \sum_{n \le x} \Lambda(n), \quad \theta(x) = \sum_{p \le x} \log p.$$

By the definition of $\Lambda(n)$,

$$\psi(x) - \theta(x) = \sum_{p^k \le x, k \ge 2} \log p = \sum_{p \le \sqrt{x}} \log p \sum_{k \le \log x / \log p} 1 \ll \sqrt{x} \log x.$$

So

$$\psi(x) = \theta(x) + O(\sqrt{x} \log x). \tag{4.1}$$

In practice, we usually investigate $\psi(x)$ instead of $\pi(x)$. The reason is that $\Lambda(n)$ is more closely related to the Riemann ζ-function (see Theorem 2.13).

4.1 Chebyshev's estimate

The first remarkable estimate for $\pi(x)$ is given by Chebyshev. He proves that the prime number theorem holds "in the sense of order". Moreover, the proof of this conclusion is quite elegant.

Theorem 4.1 (Chebyshev's estimate)**.** *For $x \geq 2$, we have*

$$\psi(x) \asymp x.$$

Proof. We begin with the sum

$$S(x) = \sum_{mn \leq x} \Lambda(m).$$

We will give two different expressions of $S(x)$:

i). $S(x) = x \log x - x + O(\log x)$.

ii). $S(x) = \sum_{n \leq x} \psi\left(\frac{x}{n}\right)$.

In fact, we have

$$S(x) = \sum_{mn \leq x} \Lambda(m) = \sum_{d \leq x} \sum_{mn=d} \Lambda(m) = \sum_{d \leq x} \log d.$$

From this and Theorem 3.5, we deduce the first expression. Changing the order of summation, we obtain the second expression:

$$S(x) = \sum_{mn \leq x} \Lambda(m) = \sum_{n \leq x} \sum_{m \leq x/n} \Lambda(m) = \sum_{n \leq x} \psi\left(\frac{x}{n}\right).$$

Therefore, we have

$$\psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) - \cdots = S(x) - 2S\left(\frac{x}{2}\right) = x \log 2 + O(\log x). \tag{4.2}$$

By the monotonicity of $\psi(x)$, we infer two inequalities from (4.2):

$$\psi(x) - \psi\left(\frac{x}{2}\right) \leq x \log 2 + O(\log x) \leq \psi(x).$$

The second inequlity has already given the desired lower bound for $\psi(x)$. Repeatedly applying the first inequality, we obtain the desired upper bound:

$$\psi(x) \leq \psi\left(\frac{x}{2}\right) + x \log 2 + O(\log x) \leq \cdots \leq x \log 4 + O(\log^2 x) \ll x.$$

□

Corollary 4.2. *For $x \geq 2$, we have*

$$\pi(x) = \frac{\theta(x)}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

Proof. By partial summation, we have

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(u)}{u \log^2 u}\,\mathrm{d}u + O(1).$$

By Chebyshev's estimate and (4.1), we have

$$\theta(x) \asymp x.$$

Thus

$$\begin{aligned}\int_2^x \frac{\theta(u)}{u \log^2 u}\,\mathrm{d}u &\ll \int_2^x \frac{1}{\log^2 u}\,\mathrm{d}u = \int_2^{\sqrt{x}} \frac{1}{\log^2 u}\,\mathrm{d}u + \int_{\sqrt{x}}^x \frac{1}{\log^2 u}\,\mathrm{d}u \\ &\ll \sqrt{x} + \frac{x}{\log^2 x} \ll \frac{x}{\log^2 x}.\end{aligned}$$

The desired result follows. □

Corollary 4.3. *For $x \geq 2$, we have*

$$\pi(x) \asymp \frac{x}{\log x}.$$

Corollary 4.4. *The following statements are equivalent:*

$$\begin{aligned}\pi(x) &\sim \frac{x}{\log x}, \quad x \to +\infty, \\ \psi(x) &\sim x, \quad x \to +\infty, \\ \theta(x) &\sim x, \quad x \to +\infty.\end{aligned}$$

Corollary 4.5. *There exists some constant $A > 1$, such that for sufficiently large x, the interval $[x, Ax]$ contains at least one prime number.*

Remark. Actually, we can prove that for any integer $n > 1$, the interval $[n, 2n]$ always contains a prime number. This conclusion is known as **Bertrand's postulate**. Assuming the prime number theorem, we can show that there exists a function $\Delta(x)$ with $\Delta(x) = o(x)$ as $x \to +\infty$ such that the interval $[x, x + \Delta(x)]$ always contains a prime for sufficiently large x.

4.2 Mertens' theorem

Some weighted sum over primes is easier to investigate than the prime counting functions $\pi(x)$. This is the case for the sum evaluated in the following theorem:

Theorem 4.6 (Mertens' first theorem)**.** *For $x \geq 2$, we have*

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1) \tag{4.3}$$

and

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1). \tag{4.4}$$

Proof. We first show that these two assertions are equivalent. In fact, we have

$$\begin{aligned} \sum_{n \leq x} \frac{\Lambda(n)}{n} - \sum_{p \leq x} \frac{\log p}{p} = \sum_{\substack{p^k \leq x \\ k \geq 2}} \frac{\log p}{p^k} &\leq \sum_{p \leq \sqrt{x}} \log p \sum_{k=2}^{\infty} \frac{1}{p^k} \\ &\ll \sum_{p \leq \sqrt{x}} \frac{\log p}{p^2} \ll 1. \end{aligned}$$

So it suffices to prove (4.4). On the one hand, we have

$$\begin{aligned} \sum_{n \leq x} \Lambda(n) \left[\frac{x}{n}\right] &= \sum_{n \leq x} \Lambda(n) \sum_{m \leq x/n} 1 = \sum_{m \leq x} \sum_{n \leq x/m} \Lambda(n) = \sum_{d \leq x} \sum_{mn = d} \Lambda(n) \\ &= \sum_{d \leq x} \log d = x \log x + O(x). \end{aligned}$$

On the other hand,

$$\sum_{n \leq x} \Lambda(n) \left[\frac{x}{n}\right] = x \sum_{n \leq x} \frac{\Lambda(n)}{n} + O(\psi(x)) = x \sum_{n \leq x} \frac{\Lambda(n)}{n} + O(x).$$

Thus

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1).$$

□

By partial summation, we can derive the following asymptotic formula from Mertens' first theorem.

Theorem 4.7. *There exists a constant C such that for any $x \geq 30$, we have*

$$\sum_{p\leq x}\frac{1}{p} = \log\log x + C + O\left(\frac{1}{\log x}\right).$$

Proof. Write

$$\sum_{p\leq x}\frac{1}{p} = \frac{1}{2} + \sum_{2<p\leq x}\frac{1}{p} = \frac{1}{2} + \int_2^x \frac{1}{\log u}\,\mathrm{d}\sum_{p\leq u}\frac{\log p}{p}.$$

Then the desired result follows from Theorem 4.6 and partial summation. □

Theorem 4.8. *There exists a constant $c > 0$ such that for any $x \geq 30$, we have*

$$\prod_{p\leq x}\left(1-\frac{1}{p}\right) = \frac{C}{\log x}\left\{1 + O\left(\frac{1}{\log x}\right)\right\}.$$

Proof. Since

$$\prod_{p\leq x}\left(1-\frac{1}{p}\right) = \exp\left\{\sum_{p\leq x}\log\left(1-\frac{1}{p}\right)\right\}, \tag{4.5}$$

it suffices to give the asymptotic formula of

$$\sum_{p\leq x}\log\left(1-\frac{1}{p}\right).$$

By Taylor's formula, we have

$$\log\left(1-\frac{1}{p}\right) = -\frac{1}{p} + O\left(\frac{1}{p^2}\right).$$

So the series

$$\sum_{p=1}^{\infty}\left\{\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right\}$$

is convergent and we have

$$\sum_{p\leq x}\left\{\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right\} = c_1 + \sum_{p>x}\left\{\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right\} = c_1 + O\left(\frac{1}{x}\right)$$

where

$$c_1 = \sum_{p=1}^{\infty} \left\{ \log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right\}. \tag{4.6}$$

Therefore,

$$\begin{aligned} \sum_{p \le x} \log\left(1 - \frac{1}{p}\right) &= -\sum_{p \le x} \frac{1}{p} + \sum_{p \le x} \left\{ \log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right\} \\ &= -\log\log x - c_2 + O\left(\frac{1}{\log x}\right) + c_1 + O\left(\frac{1}{x}\right) \\ &= -\log\log x + c_3 + O\left(\frac{1}{\log x}\right) \end{aligned} \tag{4.7}$$

where c_1 is given by (4.6), c_2 is the constant in Theorem 4.7 and

$$c_3 = c_1 - c_2.$$

Substituting (4.7) into (4.5), we obtain

$$\begin{aligned} \prod_{p \le x} \left(1 - \frac{1}{p}\right) &= \exp\left\{ \sum_{p \le x} \log\left(1 - \frac{1}{p}\right) \right\} \\ &= \exp\left\{ -\log\log x + c_3 + O\left(\frac{1}{\log x}\right) \right\} \\ &= \frac{C}{\log x} \left\{ 1 + O\left(\frac{1}{\log x}\right) \right\} \end{aligned}$$

where $C = e^{c_3}$. □

Remark. The constant C in Theorem 4.8 can be explicitly computed. It can be proved that $C = e^{-\gamma}$. This asymptotic formula is called Mertens' second theorem.

Theorem 4.9. *We have*

$$\liminf_{x \to +\infty} \frac{\pi(x)}{x/\log x} \le 1 \le \limsup_{x \to +\infty} \frac{\pi(x)}{x/\log x}.$$

As a consequence, if

$$\pi(x) \sim \frac{cx}{\log x}$$

for some constant c, then $c = 1$.

Proof. We only prove

$$\limsup_{x\to+\infty}\frac{\pi(x)}{x/\log x}\geq 1.$$

The other inequality can be proved similarly. Let

$$U=\limsup_{x\to+\infty}\frac{\pi(x)}{x/\log x}.$$

Then for any $\varepsilon>0$, there exists some $x_0>0$ such that for any $x>x_0$, we have

$$\pi(x)\leq (U+\varepsilon)\frac{x}{\log x}$$

Therefore, by partial summation, for $x>x_0$, we have

$$\begin{aligned}\sum_{p\leq x}\frac{1}{p}&=\sum_{p\leq x_0}\frac{1}{p}+\sum_{x_0<p\leq x}\frac{1}{p}\\&=O_\varepsilon(1)+\int_{x_0}^{x}\frac{1}{t}\,\mathrm{d}\pi(t)\\&=O_\varepsilon(1)+\frac{\pi(x)}{x}-\frac{\pi(x_0)}{x_0}+\int_{x_0}^{x}\frac{\pi(t)}{t^2}\,\mathrm{d}t\\&=O_\varepsilon(1)+O\left((U+\varepsilon)\int_{x_0}^{x}\frac{1}{t\log t}\,\mathrm{d}t\right)\\&\leq (U+\varepsilon)\log\log x+O_\varepsilon(1).\end{aligned}$$

By Theorem 4.7, we have $U+\varepsilon\geq 1$ and hence $U\geq 1$ since ε is arbitrary. □

4.3 Average orders of $\omega(n)$

In this section, we consider the average orders of $\omega(n)$.

Theorem 4.10. *There exist some constant C such that for any $x\geq 30$,*

$$\sum_{n\leq x}\omega(n)=x\log\log x+Cx+O\left(\frac{x}{\log x}\right).$$

The constant C is the same with the constant C in Theorem 4.7.

Proof. The proof is fairly straightforward. We have

$$\begin{aligned}\sum_{n\le x}\omega(n) &= \sum_{n\le x}\sum_{p\mid n}1=\sum_{p\le x}\sum_{\substack{n\le x\\ p\mid n}}1=\sum_{p\le x}\left[\frac{x}{p}\right]\\ &= x\sum_{p\le x}\frac{1}{p}+O\left(\pi(x)\right)\\ &= x\left(\log\log x+C+O\left(\frac{1}{\log x}\right)\right)+O\left(\frac{x}{\log x}\right)\\ &= x\log\log x+Cx+O\left(\frac{x}{\log x}\right),\end{aligned}$$

where we have used Theorem 4.7 and Chebyshev's estimate in the third "=". □

Remark. The same conclusion holds for $\Omega(n)$ (but not with the same constant C). In fact, we can also replace $\omega(n)$ by $\Omega(n)$ in the next two theorems.

Next we investigate the second moment of $\omega(n)$.

Theorem 4.11. *For $x\ge 30$, we have*

$$\sum_{n\le x}\omega^2(n)=x\left(\log\log x\right)^2+O\left(x\log\log x\right).$$

Proof. We have

$$\begin{aligned}\sum_{n\le x}\omega^2(n) &= \sum_{n\le x}\left(\sum_{p_1\mid n}1\right)\left(\sum_{p_2\mid n}1\right)=\sum_{p_1\le x}\sum_{p_2\le x}\sum_{\substack{n\le x\\ [p_1,p_2]\mid n}}1\\ &= \sum_{\substack{p_1\le x\ p_2\le x\\ p_1\ne p_2}}\left[\frac{x}{p_1p_2}\right]+\sum_{p\le x}\sum_{\substack{n\le x\\ p\mid n}}1.\end{aligned}$$

For the second term, we have

$$\sum_{p\le x}\sum_{\substack{n\le x\\ p\mid n}}1=\sum_{n\le x}\omega(n)=O\left(x\log\log x\right)$$

by Theorem 4.10. For the first term, we have

$$\begin{aligned}\sum_{\substack{p_1\le x\\ p_1\neq p_2}}\sum_{p_2\le x}\left[\frac{x}{p_1p_2}\right] &= \sum_{p_1\le x}\sum_{p_2\le x}\left[\frac{x}{p_1p_2}\right]-\sum_{p\le x}\sum_{p^2|n}\left[\frac{x}{p^2}\right]\\ &= \sum_{p_1}\sum_{\substack{p_2\\ p_1p_2\le x}}\left[\frac{x}{p_1p_2}\right]+O(x)\\ &= x\sum_{p_1}\sum_{\substack{p_2\\ p_1p_2\le x}}\frac{1}{p_1p_2}+O\left(\sum_{p\le x}\pi\left(\frac{x}{p}\right)\right).\end{aligned}$$

By Theorem 4.7, the O-term

$$\ll x\sum_{p\le x}\frac{1}{p}\ll x\log\log x.$$

For the main term, we notice that

$$\left(\sum_{p\le\sqrt{x}}\frac{1}{p}\right)^2\le\sum_{p_1}\sum_{\substack{p_2\\ p_1p_2\le x}}\frac{1}{p_1p_2}\le\left(\sum_{p\le x}\frac{1}{p}\right)^2.$$

Both sides of the above inequality are

$$(\log\log x+O(1))^2=(\log\log x)^2+O\left(x\log\log x\right).$$

So the same asymptotic formula holds for

$$\sum_{p_1}\sum_{\substack{p_2\\ p_1p_2\le x}}\frac{1}{p_1p_2}.$$

Combining all the above results, we get the desired conclusion. □

Theorem 4.12. *Let $\varepsilon>0$ be a fixed number. For sufficiently large x, the number of integers n with $1\le n\le x$ such that*

$$|\omega(n)-\log\log x|\ge(\log\log x)^{\frac{1}{2}+\varepsilon}$$

is

$$O\left(\frac{x}{(\log\log x)^{2\varepsilon}}\right).$$

Proof. Let

$$\mathcal{A}(x) = \left\{ n \le x \,:\, |\omega(n) - \log\log x| \ge (\log\log x)^{\frac{1}{2}+\varepsilon} \right\}.$$

Then we have

$$\begin{aligned}
|\mathcal{A}(x)|\,(\log\log x)^{1+2\varepsilon} &\le \sum_{n\in\mathcal{A}(x)} |\omega(n) - \log\log x|^2 \\
&= \sum_{n\le x} |\omega(n) - \log\log x|^2 \\
&= \sum_{n\le x} \omega^2(n) - 2\log\log x \sum_{n\le x} \omega(n) + x(\log\log x)^2 \\
&= \left\{ x(\log\log x)^2 + O(x\log\log x) \right\} \\
&\quad - 2\log\log x\,(x\log\log x + O(x)) + x(\log\log x)^2 \\
&= O(x\log\log x).
\end{aligned}$$

The desired result follows. □

From the point of view of probability theory, Theorem 4.12 is nothing but a direct application of the Chebyshev inequality. We regard $\omega(n)$ as a random variable on the probability space $\mathbb{N} \cap [1, x]$ equipped with the uniform probability. Then Theorem 4.10 implies that the expectation of ω is $\log\log x$ asymptotically. Theorem 4.11 implies that the variance of ω is $O(\log\log x)$. So by the Chebyshev inequality, for any $D > 0$, the probability of the event "$|\omega(n) - \log\log x| > D$" does not exceed

$$\frac{\mathrm{Var}(\omega)}{D^2} = O\left(\frac{\log\log x}{D^2}\right).$$

To obtain a non-trivial conclusion from this inequality, we should take $D > (\log\log x)^{1/2}$. By taking $D = (\log\log)^{1/2+\varepsilon}$, we obtain Theorem 4.12.

A more interesting question is to investigate the asymptotic behaviour of the distribution of an arithmetic function as a random variable. For example, we have the "central limit theorem" for $\omega(n)$:

Theorem 4.13 (Erdös–Kac, 1939)**.** *We have uniformly for $x \ge 30$, $y \in \mathbb{R}$ that*

$$\frac{1}{x}\left|\left\{ n \le x \,:\, \omega(n) \le \log\log x + y(\log\log x)^{1/2} \right\}\right| = \Phi(y) + O\left(\frac{1}{(\log\log x)^{1/2}}\right),$$

where $\Phi(y)$ is the normal distribution function

$$\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2}\,\mathrm{d}t.$$

Proof. Omitted. □

4.4 Equivalent conditions of the prime number theorem

In this section, we give some propositions equivalent to the prime number theorem. Let $M(x)$ denote the **Mertens function** defined by

$$M(x) = \sum_{n \le x} \mu(n).$$

We will show that $M(x) = o(x)$ is equivalent to $\psi(x) \sim x$. Hence by Corollary 4.4, it is equivalent to the prime number theorem. We begin with a simple lemma.

Lemma 4.14. *Let*

$$H(x) = \sum_{n \le x} \mu(n) \log n.$$

Then $H(x) = o(x \log x)$ is equivalent to $M(x) = o(x)$.

Proof. By partial summation, we have

$$H(x) = \int_1^x \log u \, \mathrm{d}\pi(u) = \pi(x) \log x - \int_1^x \frac{\pi(u)}{u} \, \mathrm{d}u \quad = \pi(x) \log x + O(x).$$

Therefore, we have

$$\frac{H(x)}{x \log x} = \frac{M(x)}{x} + O\left(\frac{1}{\log x}\right).$$

The desired result follows. □

Theorem 4.15. *The prime number theorem is equivalent to $M(x) = o(x)$.*

Proof. Suppose that $\psi(x) = o(x)$. By Theorem 2.10 and the Möbius inversion formula (Theorem 2.6), we have

$$-\mu(n) \log n = \sum_{d \mid n} \mu(d) \Lambda\left(\frac{n}{d}\right).$$

By Dirichlet's hyperbola method (Theorem 3.11), we see for any $1 \le y \le x$ that,

$$-H(x) = -\sum_{n \le x} \mu(n) \log n = \sum_{n \le y} \mu(n) \psi\left(\frac{x}{n}\right) + \sum_{n \le x/y} \Lambda(n) M\left(\frac{x}{n}\right) - M(y) \psi\left(\frac{x}{y}\right).$$

We specify $y = x/\log x$ and estimate the above three terms.

- The estimate of $M(y)\psi(x/y)$ is easy. We use the trivial bound for $M(y)$ and use Chebyshev's estimate for $\psi(x/y)$ to obtain that

$$M(y)\psi\left(\frac{x}{y}\right) \ll y \cdot \frac{x}{y} = x = o(x \log x).$$

- To estimate the second term, we trivially bound $M(x/n)$ by x/n and apply (4.4) to obtain that

$$\sum_{n \le x/y} \Lambda(n) M\left(\frac{x}{n}\right) \le x \sum_{n \le x/y} \frac{\Lambda(n)}{n} \ll x \log \frac{x}{y} = x \log\log x = o(x \log x).$$

- Finally, we estimate $\sum_{n \le y} \mu(n)\psi(x/n)$. Since $\psi(x) \sim x$, we can write

$$\psi(x) = x(1 + R(x))$$

where $R(x) \to 0$ as $x \to +\infty$. So we have

$$\sum_{n \le y} \mu(n)\psi\left(\frac{x}{n}\right) = x \sum_{n \le y} \frac{\mu(n)}{n} + O\left(x \sum_{n \le y} \frac{R(x/n)}{n}\right).$$

By Corollary 2.8, we have

$$x \sum_{n \le y} \frac{\mu(n)}{n} = O(x).$$

For the O-term, we have

$$x \sum_{n \le y} \frac{R(x/n)}{n} \le x \left(\sup_{x/y \le u \le x} |R(u)|\right) \left(\sum_{n \le y} \frac{1}{n}\right) \ll x \log x \left(\sup_{x/y \le u \le x} |R(u)|\right).$$

Since $y = \log x$, we have $x/y \to +\infty$ as $x \to +\infty$. Thus

$$\lim_{x \to +\infty} \sup_{x/y \le u \le x} |R(u)| = 0.$$

Therefore,

$$\sum_{n \le y} \mu(n)\psi\left(\frac{x}{n}\right) = o(x \log x).$$

In summary, we have deduced that $H(x) = o(x \log x)$. So by Lemma 4.14, we have $M(x) = o(x)$.

Conversely, assuming that $M(x) = o(x)$, we would like to prove $\psi(x) \sim x$. We first claim the following identity:

$$\psi(x) = x - \sum_{mn \le x} \mu(m) f(n) + O(1), \tag{4.8}$$

where

$$f(n) = \tau(n) - \log n - 2\gamma.$$

This identity can be easily deduced from the following Dirichlet convolutions:

- $\mu * \tau = \mu * \mathbb{1} * \mathbb{1} = \delta * \mathbb{1} = \mathbb{1}$.
- $\mu * \log = \Lambda$.
- $\mu * \mathbb{1} = \delta$.

Moreover, by Theorem 3.10 and Theorem 3.7, we have

$$F(x) := \sum_{n \le x} f(n) = O\left(\sqrt{x}\right). \tag{4.9}$$

By (4.8), it suffices to show

$$\sum_{mn \le x} \mu(m) f(n) = o(x).$$

We again apply Dirichlet's hyperbola method. For any $1 \le y \le x$, we have

$$\sum_{mn \le x} \mu(m) f(n) = \sum_{n \le y} \mu(n) F\left(\frac{x}{n}\right) + \sum_{n \le x/y} f(n) M\left(\frac{x}{n}\right) - M(y) F\left(\frac{x}{y}\right).$$

We estimate these three terms.

- For the first term, we apply (4.9) to obtain that
$$\sum_{n \le y} \mu(n) F\left(\frac{x}{n}\right) \ll \sqrt{x} \sum_{n \le y} \frac{1}{\sqrt{n}} \ll \sqrt{xy}. \tag{4.10}$$
- For the third term, we again use (4.9) and estimate $M(y)$ trivially to obtain that
$$M(y) F\left(\frac{x}{y}\right) \ll y \sqrt{\frac{x}{y}} = \sqrt{xy}. \tag{4.11}$$

- The main difficulty comes from the second term. We are going to show that for any $\varepsilon > 0$,
$$\left|\sum_{mn\le x} \mu(m) f(n)\right| \le \varepsilon x.$$
So we need to specify the implied constants in (4.10) and (4.11): Let $C_1 > 0$ be such that
$$\max\left(\left|\sum_{n\le y}\mu(n)F\left(\frac{x}{n}\right)\right|, \left|M(y)F\left(\frac{x}{y}\right)\right|\right) \le C_1\sqrt{xy}.$$
By Proposition 3.1 and the definition of $f(n)$, we have
$$f(n) \ll \sqrt{n}.$$
Since now we assume that $M(x) = o(x)$, we can write $M(x) = xR(x)$ with $R(x) \to 0$ as $x \to +\infty$. Therefore, we have
$$\sum_{n\le x/y} f(n)M\left(\frac{x}{n}\right) = x\sum_{n\le x/y}\frac{f(n)}{n}R\left(\frac{x}{n}\right) \le C_2 x\sqrt{\frac{x}{y}}\sup_{y\le u\le x}|R(u)|$$
for some absolute constant $C_2 > 0$.

Now we fix an arbitrarily small positive number ε. Since $R(x) \to 0$, there exists some $X_0 > 0$ such that for any $x > X_0$, we have
$$|R(x)| \le \frac{\varepsilon^2}{9C_1C_2}.$$
Set
$$y = \left(\frac{\varepsilon}{3C_1}\right)^2 x.$$
Then for any $x > (3C_1/\varepsilon)^2 X_0$, we have
$$\max\left(\left|\sum_{n\le y}\mu(n)F\left(\frac{x}{n}\right)\right|, \left|M(y)F\left(\frac{x}{y}\right)\right|\right) \le C_1\sqrt{xy} \le \frac{\varepsilon}{3}x$$
and
$$\sum_{n\le x/y} f(n)M\left(\frac{x}{n}\right) \le C_2 x\sqrt{\frac{x}{y}}\sup_{y\le u\le x}|R(u)| \le C_2 x\frac{3C_1}{\varepsilon}\cdot\frac{\varepsilon^2}{9C_1C_2} = \frac{\varepsilon}{3}x.$$
The proof is complete. □

Remark. Similarly, we can prove that the estimate

$$L(x) := \sum_{n \leq x} = o(x)$$

is equivalent to the prime number theorem.