Chapter 3

Average orders

We first introduce some notations. Let $f(x)$ and $g(x)$ be functions. We use interchangeably Landau's notation

$$f(x) = O(g(x))$$

and Vinogradov's notaion

$$f(x) \ll g(x)$$

to both mean that

$$|f(x)| \leq C|g(x)|$$

for some positive constant C. Sometimes, the implied constant C may depend on other parameters. For example, for any $A > 0$ and $x \geq 2$, we have

$$\log^A x = O(x),$$

which means there exists a positive constant C s.t.

$$\log^A x \leq Cx$$

holds for all $x \in [2, +\infty)$. However, this inequality could not hold uniformly for all $A > 0$, so the constant C depends on A. We indicate this dependence in subscript, e.g. $\log^A x = O_A(x)$ or $\log^A x \ll_A x$. Moreover, we write $f \asymp g$ to indicate that $f \ll g$ and $g \ll f$ hold simultaneously.

3.1 Introduction

We are concerned about the asymptotic behaviour of arithmetic functions. However, arithmetic functions usually have very erratic behaviour. For example, the divisor function $\tau(n) = 2$ if n is a prime number. But $\tau(n)$ could be larger than $\log^A n$ for any $A > 0$:

Proposition 3.1. *We have $\tau(n) = O_\varepsilon(n^\varepsilon)$ for arbitrary $\varepsilon > 0$. However, the estimation $\tau(n) = O(\log^A n)$ fails to hold for any $A > 0$.*

Proof. Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be the prime factorization of n. Then we have

$$\frac{\tau(n)}{n^\varepsilon} = \prod_{j=1}^{k} \frac{1+\alpha_j}{p_j^{\alpha_j \varepsilon}} = \prod_{\substack{j=1 \\ p_j \le 2^{1/\varepsilon}}}^{k} \frac{1+\alpha_j}{p_j^{\alpha_j \varepsilon}} \prod_{\substack{j=1 \\ p_j > 2^{1/\varepsilon}}}^{k} \frac{1+\alpha_j}{p_j^{\alpha_j \varepsilon}}.$$

For those p_j with $p_j > 2^{1/\varepsilon}$, we have

$$\frac{1+\alpha_j}{p_j^{\alpha_j \varepsilon}} \le \frac{1+\alpha_j}{2^{\alpha_j}} \le 1.$$

So it suffices to bound

$$\prod_{\substack{j=1 \\ p_j \le 2^{1/\varepsilon}}}^{k} \frac{1+\alpha_j}{p_j^{\alpha_j \varepsilon}}.$$

The number of p_j's with $p_j \le 2^{1/\varepsilon}$ does not exceed $2^{1/\varepsilon}$ and we can bound this part trivially by

$$\prod_{\substack{j=1 \\ p_j \le 2^{1/\varepsilon}}}^{k} \frac{1+\alpha_j}{p_j^{\alpha_j \varepsilon}} = \prod_{\substack{j=1 \\ p_j \le 2^{1/\varepsilon}}}^{k} \frac{1+\alpha_j}{e^{\alpha_j \varepsilon \log p_j}} \le \prod_{\substack{j=1 \\ p_j \le 2^{1/\varepsilon}}}^{k} \frac{1+\alpha_j}{\alpha_j \varepsilon \log p_j} \le \left(\frac{2}{\varepsilon \log 2} \right)^{2^{1/\varepsilon}}.$$

So we obtain $\tau(n) = O_\varepsilon(n^\varepsilon)$ with a possible choice of implied constant given by

$$C(\varepsilon) = \left(\frac{2}{\varepsilon \log 2} \right)^{2^{1/\varepsilon}}.$$

To show that $\tau(n) = O(\log^A n)$ fails to hold, it suffices to find a sequence $\{n_k\}$ with $n_k \to \infty$ s.t.

$$\frac{\tau(n_k)}{\log^A n_k}.$$

is unbonded. Let $m > A$ be a positive integer and we arbitrarily choose m distinct (and fixed) prime numbers $p_1, \dots, p_m$. Let

$$n_k = (p_1 \cdots p_m)^k.$$

Then we have

$$k = \frac{\log n_k}{\log(p_1 \cdots p_m)} \gg \log n_k$$

where the implied constant depends on $p_1, \dots, p_m$ but is independent of k. For this n_k, we have

$$\tau(n_k) = (k+1)^m \gg \log^m n_k.$$

Since $m > A$, it is clear that

$$\lim_{k\to\infty} \frac{\tau(n_k)}{\log^A n_k} = +\infty.$$

This completes the proof. □

We will see soon that "in average", we have $\tau(n) \approx \log n$. More precisely, we have

$$\frac{1}{N}\sum_{n\le N} \tau(n) \sim \frac{1}{N}\sum_{n\le N} \log n \quad \text{as} \quad N \to \infty.$$

This inspires us to investigate the average behaviour of an arithmetic function.

3.2 Summation formulae

In practice, we often need to consider the relationship between the sum $\sum f(n)$ with the weighted sum $\sum w(n)f(n)$ where $w(x)$ is a smooth function. On the other hand, we also need to know the relationship between the sum of a smooth function and its integral. Both of the above can be treated using the partial integration of Stieltjes integrals.

Theorem 3.2 (Partial summation)**.** *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Let*

$$A(x) = \sum_{n\le x} a_n.$$

Let $f(x)$ be a continuously differentiable function on the interval $[1, x]$. Then for any $1 \le a < b \le x$, we have

$$\sum_{a<n\le b} a_n f(n) = A(b)f(b) - A(a)f(a) - \int_a^b f'(x)A(x)\,\mathrm{d}x.$$

Proof. By the partial integration of Stieltjes integrals, we have

$$\sum_{a<n\le b} a_n f(n) = \int_a^b f(x)\,\mathrm{d}A(x) = A(x)b(x)\Big|_a^b - \int_a^b f'(x)A(x)\,\mathrm{d}x$$
$$= A(b)f(b) - A(a)f(a) - \int_a^b f'(x)A(x)\,\mathrm{d}x.$$

□

Theorem 3.3 (Euler–Maclaurin summation formula)**.** *Let $f(x)$ be a continuously differentiable function on $[a, b]$. Let $\rho(x)$ be the "saw function" defined by*

$$\rho(x) = \frac{1}{2} - \{x\}.$$

Then we have

$$\sum_{a<n\le b} f(n) = \int_a^b f(x)\,\mathrm{d}x + \rho(x)f(x)\Big|_a^b - \int_a^b \rho(x)f'(x)\,\mathrm{d}x.$$

Moreover, if $f(x)$ is twice continuously differentiable on $[a, b]$, we have

$$\sum_{a<n\le b} f(n) = \int_a^b f(x)\,\mathrm{d}x + \rho(x)f(x)\Big|_a^b - \sigma(x)f'(x)\Big|_a^b + \int_a^b \sigma(x)f''(x)\,\mathrm{d}x,$$

where

$$\sigma(x) = \int_0^x \rho(t)\,\mathrm{d}t.$$

Proof. The sum of $f(n)$ can be written as the Stieltjes integral.

$$\sum_{a<n\le b} f(n) = \int_a^b f(x)\,\mathrm{d}[x] = \int_a^b f(x)\,\mathrm{d}\left(x + \rho(x)\right) = \int_a^b f(x)\,\mathrm{d}x + \int_a^b f(x)\,\mathrm{d}\rho(x).$$

By partial integration, we have

$$\int_a^b f(x)\,\mathrm{d}\rho(x) = \rho(x)f(x)\Big|_a^b - \int_a^b \rho(x)f'(x)\,\mathrm{d}x.$$

This proves the first assertion. If $f(x)$ is twice continuously differentiable, we can continue to use partial integration for the last integral to obtain the second assertion.

□

Theorem 3.4. *For $x \geq 2$, we have*

$$\sum_{n\leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right),$$

where

$$\gamma = \lim_{N\to+\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \log N \right)$$

is the Euler constant.

Proof. We apply Theorem 3.3 for $f(x) = 1/x$ to obtain that

$$\begin{aligned}\sum_{n\leq x} \frac{1}{n} &= 1 + \sum_{1<n\leq x} \frac{1}{n} \\ &= 1 + \int_1^x \frac{1}{t}\,\mathrm{d}t + \frac{\rho(x)}{x} - \rho(1) + \int_1^x \frac{\rho(t)}{t^2}\,\mathrm{d}t \\ &= \log x + \frac{1}{2} + \int_1^\infty \frac{\rho(t)}{t^2}\,\mathrm{d}t + O\left(\frac{1}{x}\right).\end{aligned}$$

Let $x \to +\infty$, we find that the constant

$$\frac{1}{2} + \int_1^\infty \frac{\rho(t)}{t^2}\,\mathrm{d}t = \lim_{x\to+\infty} \left(\sum_{n\leq x} \frac{1}{n} - \log x \right) = \gamma.$$

□

Theorem 3.5. *For $x \geq 2$, we have*

$$\sum_{n\leq x} \log n = x\log x - x + O(\log x).$$

Proof. By the Euler–Maclaurin summation formula, we have

$$\begin{aligned}\sum_{n\leq x} \log n &= \int_1^x \log t\,\mathrm{d}t + \rho(x)\log x - \rho(1)\log 1 - \int_1^x \frac{\rho(t)}{t} dt \\ &= x\log x - x + O(\log x).\end{aligned}$$

□

Some summation formulae can not simply verified by a combinatorial argument, e.g. the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

Such formulae usually appear in the automorphic theory.

3.3 Average orders of some arithmetic functions

We use the Euler–Maclaurin summation formula to investigate the average orders of some arithmetic functions.

Theorem 3.6. *For $x \geq 2$, we have*

$$\sum_{n \leq x} \tau(n) = x \log x + O(x)$$

Proof. We exchange the order of summations to obtain that

$$\begin{aligned} \sum_{n \leq x} \tau(n) &= \sum_{n \leq x} \sum_{d \mid n} 1 = \sum_{d \leq x} \sum_{\substack{n \leq x \\ d \mid n}} 1 = \sum_{d \leq x} \left[\frac{x}{d} \right] \\ &= \sum_{d \leq x} \left(\frac{x}{d} - \left\{ \frac{x}{d} \right\} \right) = x \sum_{d \leq x} \frac{1}{d} + O(x). \end{aligned} \tag{3.1}$$

By Theorem 3.4, we have

$$\sum_{d \leq x} \frac{1}{d} = \log x + O(1). \tag{3.2}$$

Substituting (3.2) into (3.1), we get the desired result. □

Remark. In fact, we have not exactly found the correct main term. In next section, we will show that

$$\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

Theorem 3.7. *For $x \geq 2$, we have*

$$\sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x \log x).$$

Proof. By Theorem 2.9, we have

$$\begin{aligned}\sum_{n\le x}\varphi(n) &= \sum_{n\le x}\sum_{d|n}\mu(d)\frac{n}{d} = \sum_{d\le x}\frac{\mu(d)}{d}\sum_{\substack{n\le x\\ d|n}} n\\ &= \frac{1}{2}\sum_{d\le x}\mu(d)\left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right]+1\right)\\ &= \frac{1}{2}\sum_{d\le x}\mu(d)\left(\frac{x}{d}+O(1)\right)^2\\ &= \frac{x^2}{2}\sum_{d\le x}\frac{\mu(d)}{d^2}+O\left(x\sum_{d\le x}\frac{1}{d}\right).\end{aligned}$$

By Theorem 2.12, we see that

$$\sum_{d\le x}\frac{\mu(d)}{d^2} = \sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}+O\left(\frac{1}{x}\right) = \frac{1}{\zeta(2)}+O\left(\frac{1}{x}\right) = \frac{6}{\pi^2}+O\left(\frac{1}{x}\right).$$

By Theorem 3.4, we have

$$x\sum_{d\le x}\frac{1}{d} \ll x\log x.$$

The desired result follows. □

Theorem 3.8. *For $x \ge 2$, we have*

$$\sum_{n\le x}\sigma(n) = \frac{\pi^2}{12}x^2+O(x\log x).$$

Proof. We have

$$\begin{aligned}\sum_{n\le x}\sigma(n) &= \sum_{n\le x}\sum_{d|n}\frac{n}{d} = \sum_{d\le x}\frac{1}{d}\sum_{\substack{n\le x\\ d|n}} = \frac{1}{2}\sum_{d\le x}\left[\frac{x}{d}\right]\left(\left[\frac{x}{d}\right]+1\right)\\ &= \frac{1}{2}\sum_{d\le x}\left(\frac{x}{d}+O(1)\right)^2 = \frac{x^2}{2}\sum_{d\le x}\frac{1}{d^2}+O\left(x\sum_{d\le x}\frac{1}{d}\right).\end{aligned}$$

We have

$$\sum_{d\le x}\frac{1}{d^2} = \zeta(2)+O\left(\frac{1}{x}\right) = \frac{\pi^2}{6}+O\left(\frac{1}{x}\right)$$

and

$$x\sum_{d\le x}\frac{1}{d}\ll x\log x.$$

The desired result follows. □

Theorem 3.9. *Let $Q(x)$ denote the number of square-free integers not exceeding x. Then for $x\ge 2$, we have*

$$Q(x)=\frac{6}{\pi^2}x+O(\sqrt{x}).$$

Proof. Let $g(n)$ denote the characteristic functions of square-free numbers, i.e.

$$g(n)=\begin{cases}1, & n \text{ is square free}\\ 0, & \text{otherwise.}\end{cases}$$

Then

$$g(n)=\sum_{d^2|n}\mu(d).$$

So

$$\begin{aligned}Q(x)=\sum_{n\le x}g(n)=\sum_{n\le x}\sum_{d^2|n}\mu(d) &=\sum_{d\le\sqrt{x}}\mu(d)\sum_{\substack{n\le x\\ d^2|n}}=\sum_{d\le\sqrt{x}}\mu(d)\left[\frac{x}{d^2}\right]\\ =x\sum_{d\le\sqrt{x}}\frac{\mu(d)}{d^2}+O(\sqrt{x}) &=\frac{x}{\zeta(2)}+O(\sqrt{x})=\frac{6}{\pi^2}x+O(\sqrt{x}).\end{aligned}$$

□

Remark. The error terms in Theorem 3.7 – Theorem 3.9 can be improved (Walfisz, 1963):

$$\begin{aligned}\sum_{n\le x}\varphi(n) &= \frac{3}{\pi^2}x^2+O\left(x(\log x)^{2/3}(\log\log x)^{4/3}\right).\\ \sum_{n\le x}\sigma(n) &= \frac{\pi^2}{12}x^2+O\left(x(\log x)^{2/3}\right).\\ Q(x) &= \frac{6}{\pi^2}x+O\left(\sqrt{x}\exp\left\{-c\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right\}\right).\end{aligned}$$

3.4 Dirichlet's hyperbola method

We improve the result of Theorem 3.6.

Theorem 3.10. *For $x \geq 2$, we have*

$$\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

Proof. We have

$$\sum_{d \leq x} \tau(d) = \sum_{d \leq x} \sum_{mn=d} 1 = \sum_{mn \leq x} 1.$$

By symmetry, we have

$$\begin{aligned} \sum_{mn \leq x} 1 &= \sum_{m \leq \sqrt{x}} \sum_{n \leq \frac{x}{m}} 1 + \sum_{n \leq \sqrt{x}} \sum_{m \leq \frac{x}{n}} 1 - \sum_{n \leq \sqrt{x}} \sum_{m \leq \sqrt{x}} 1 \\ &= 2 \sum_{n \leq \sqrt{x}} \left[\frac{x}{n}\right] - [\sqrt{x}]^2. \end{aligned} \tag{3.3}$$

We apply Theorem 3.4 to obtain that

$$2 \sum_{n \leq \sqrt{x}} \left[\frac{x}{n}\right] = 2x \sum_{n \leq \sqrt{x}} \frac{1}{n} + O(\sqrt{x}) = x \log x + 2\gamma + O\left(\sqrt{x}\right).$$

Substituting this into (3.3), we get the desired result. □

The trick used in the proof of Theorem 3.10 is called **Dirichlet's hyperbola method**. We summarize its general form as follows:

Theorem 3.11 (Dirichlet's hyperbola method)**.** *Let f, g be two arithmetic functions. Let F, G be their summatory functions respectively, i.e.*

$$F(x) = \sum_{n \leq x} f(n), \quad G(x) = \sum_{n \leq x} g(n).$$

Then for any positive number a and b with $ab = x$, we have

$$\sum_{n \leq x} (f * g)(n) = \sum_{n \leq a} f(n)G(x/n) + \sum_{n \leq b} g(n)F(x/n) - F(a)G(b).$$

Proof. Since $ab = x$, we have

$$\begin{aligned}\sum_{n\le x}(f*g)(n) &= \sum_{n\le x}\sum_{mk=n} f(m)g(k) = \sum_{\substack{m \quad k\\ mk\le x}}\sum f(m)g(k)\\ &= \sum_{\substack{m\le a \quad k\\ mk\le x}}\sum f(m)g(k) + \sum_{\substack{m \quad k\le b\\ mk\le x}}\sum f(m)g(k) - \sum_{\substack{m\le a \quad k\le b\\ mk\le x}}\sum f(m)g(k)\\ &= \sum_{m\le a} f(m)G(x/m) + \sum_{k\le b} g(k)F(x/k) - F(a)G(b).\end{aligned}$$

□

This trick will be useful when we discuss equivalent forms of the prime number theorem.

3.5 Dirichlet's divisor problem

Let

$$\Delta(x) = \sum_{n\le x}\tau(n) - x(\log x + 2\gamma - 1).$$

Let α be the infimum of the set of exponents ξ such that

$$\Delta(x) \ll x^{\xi}.$$

Theorem 3.10 implies that $\alpha \le 1/2$. The exact value of α remains unknown and it is generally conjectured that $\alpha = 1/4$. In 1915, Hardy and Landau proved independently that $\Delta(x)$ is not $o(x^{1/4})$, which implies that $\alpha \ge 1/4$. The best upper bound known to date is given by Huxley in 1993:

$$\alpha \le 23/73 = 0.31506849\ldots$$

This problem is called **Dirichlet's divisor problem**. We will give the proof of the following result:

Theorem 3.12 (Voronoï,1903)**.** *For $x \ge 2$, we have*

$$\sum_{n\le x}\tau(n) = x\log x + (2\gamma - 1)x + O\left(x^{1/3}\log x\right).$$

The method we will use is due to van der Corput. More precisely, our proof is based on estimations for the so-called trigonometric sums:

$$\sum_{a<n\le b} e(f(n))$$

where $f(x)$ is a "well-behaved" real-valued function and $e(x) = e^{2\pi i x}$. This method is basic but quite effective in analytic number theory. We will use the following theorem without proof.

Theorem 3.13 (van der Corput)**.** *Let $b-a \ge 1$. Let $f(x)$ be a real function on $[a,b]$ such that $\Lambda \le f''(x) \le \eta\Lambda$ with $\Lambda > 0$, $\eta \ge 1$. Then*

$$\sum_{a<n\le b} e\left(f(n)\right) \ll \eta\Lambda^{1/2}(b-a) + \Lambda^{-1/2}.$$

Proof. See Corollary 8.13 in the book *Analytic Number Theory* by Iwaniec and Kowalski. □

Proof of Theorem 3.12. We can use the Euler–Maclaurin summation formula to deduce the following refined version of Theorem 3.4:

$$\sum_{n\le x} \frac{1}{n} = \log x + \gamma + \frac{\rho(x)}{x} + O\left(\frac{1}{x^2}\right). \tag{3.4}$$

Then we repeat the process in the proof of Theorem 3.10 to obtain that

$$\begin{aligned}\sum_{n\le x} \tau(n) &= x\log x + (2\gamma-1)x + \sqrt{x} - 2\sum_{n\le\sqrt{x}} \left\{\frac{x}{n}\right\} + O(1) \\ &= x\log x + (2\gamma-1)x + 2\sum_{n\le\sqrt{x}} \rho\left(\frac{x}{n}\right) + O(1).\end{aligned}$$

So it suffices to prove

$$\sum_{n\le\sqrt{x}} \rho\left(\frac{x}{n}\right) \ll x^{1/3}\log x.$$

To apply the van der Corput method, we need to expand $\rho(x/d)$ as a Fourier series. However, the Fourier series of $\rho(t)$ is not absolutely convergent since $\rho(t)$ is not continuous. We overcome this difficulty by replacing $\rho(t)$ by

$$\bar{\rho}_\delta(t) := \frac{1}{2\delta}\int_{-\delta}^{\delta} \rho(t+u)\,\mathrm{d}u.$$

Here $0 < \delta < 1/2$ is a small quantity, which will be specified later. To investigate the difference between ρ and $\bar{\rho}_\delta$, we introduce

$$h(t) = \left|\rho(t) - \bar{\rho}_\delta(t)\right|.$$

It is not hard to see that both $\bar{\rho}_\delta(t)$ and $h(t)$ are Lipschitz continuous. So their Fourier series are absolutely convergent. We can calculate their Fourier series:

$$\begin{aligned} \bar{\rho}_\delta(t) &= \sum_{k=1}^{\infty} a_k \sin(2\pi k x), & a_k &= \frac{1}{2\delta\pi^2 k^2}\sin(2\delta\pi k),\\ h(t) &= \frac{\delta}{2} + \sum_{k=1}^{\infty} b_k \cos(2\pi k x), & b_k &= \frac{1}{\delta\pi^2 k^2}\sin^2(\delta\pi k). \end{aligned}$$

Then we have

$$|a_k| + |b_k| \ll \min\left(\frac{1}{k}, \frac{1}{\delta k^2}\right) = \begin{cases} 1/k, & k \le 1/\delta,\\ 1/(\delta k^2), & k > 1/\delta. \end{cases} \tag{3.5}$$

To make the full use of Theorem 3.13, we apply the following dyadic decomposition:

$$\sum_{n \le \sqrt{x}} \rho\left(\frac{x}{n}\right) \ll (\log x) \sup_{1 \le y \le \sqrt{x}} |T(y)|$$

where

$$T(y) = \sum_{y/2 < n \le y} \rho\left(\frac{x}{n}\right).$$

So it suffices to prove

$$T(y) \ll x^{1/3}$$

uniformly for $y \in [1, \sqrt{x}]$.

We have

$$\left| T(y) - \sum_{y/2 < n \le y} \bar{\rho}_\delta\left(\frac{x}{n}\right)\right| \le \sum_{y/2 < n \le y} h\left(\frac{x}{n}\right).$$

So

$$|T(y)| \le \left|\sum_{y/2 < n \le y} \bar{\rho}_\delta\left(\frac{x}{n}\right)\right| + \left|\sum_{y/2 < n \le y} h\left(\frac{x}{n}\right)\right|$$

We substitute the Fourier expansions to obtain that

$$\begin{aligned}|T(y)| &\le \left|\sum_{y/2<n\le y}\sum_{k=1}^{\infty} a_k \sin\left(2\pi k\frac{x}{n}\right)\right| + \left|\frac{\delta y}{4} + \sum_{y/2<n\le y}\sum_{k=1}^{\infty} b_k \cos\left(2\pi k\frac{x}{n}\right)\right| \\ &\ll \delta y + \sum_{k=1}^{\infty}|a_k|\left|\sum_{y/2<n\le y}\sin\left(\frac{2\pi kx}{n}\right)\right| + \sum_{k=1}^{\infty}|b_k|\left|\sum_{y/2<n\le y}\cos\left(\frac{2\pi kx}{n}\right)\right|.\end{aligned}$$

Now we apple Theorem 3.13 with $f(t) = xk/t$ to obtain that

$$\left|\sum_{y/2<n\le y} e\left(\frac{xk}{n}\right)\right| \ll y\frac{(xk)^{1/2}}{y^{3/2}} + \frac{y^{3/2}}{(xk)^{1/2}} \asymp \left(\frac{xk}{y}\right)^{\frac{1}{2}} + \left(\frac{y^3}{xk}\right)^{\frac{1}{2}}. \tag{3.6}$$

Certainly, the same estimate holds for

$$\sum_{y/2<n\le y}\sin\left(\frac{2\pi kx}{n}\right) \quad \text{and} \quad \sum_{y/2<n\le y}\cos\left(\frac{2\pi kx}{n}\right).$$

Finally, we substitute the estimates (3.5) and (3.6) into $T(y)$ to obtain that

$$\begin{aligned}T(y) &\ll \delta y + \sum_{k=1}^{\infty}(|a_k|+|b_k|)\left\{\left(\frac{xk}{y}\right)^{\frac{1}{2}} + \left(\frac{y^3}{xk}\right)^{\frac{1}{2}}\right\} \\ &\ll \delta y + \sum_{k\le 1/\delta}\frac{1}{k}\left\{\left(\frac{xk}{y}\right)^{\frac{1}{2}} + \left(\frac{y^3}{xk}\right)^{\frac{1}{2}}\right\} + \sum_{k>1/\delta}\frac{1}{\delta k^2}\left\{\left(\frac{xk}{y}\right)^{\frac{1}{2}} + \left(\frac{y^3}{xk}\right)^{\frac{1}{2}}\right\} \\ &\ll \delta y + \frac{x^{1/2}}{y^{1/2}}\cdot\frac{1}{\delta^{1/2}} + \frac{y^{3/2}}{x^{1/2}} + \frac{x^{1/2}}{y^{1/2}}\cdot\frac{1}{\delta}\cdot\delta^{1/2} + \frac{y^{3/2}}{x^{1/2}}\cdot\frac{1}{\delta}\cdot\delta^{3/2} \\ &\ll \delta y + \left(\frac{x}{\delta y}\right)^{\frac{1}{2}} + \left(\frac{y^3}{x}\right)^{\frac{1}{2}}.\end{aligned}$$

Since $y \le \sqrt{x}$, the third term is admissible. To balance the first two terms, we specify $\delta = x^{1/3}/y$. Then we find that the first two terms are bounded by $x^{1/3}$. One may notice that the above argument is valid only when $y > 2x^{1/3}$ (i.e. $\delta < 1/2$). However, in the complementary case, the estimate

$$T(y) \ll x^{1/3}$$

holds trivially. So we complete the proof.