Homework 3

(due Friday, March 31, 2023)

1. Let $\tau(n)$ denote the divisor function. Prove the following assertions:

   1). For $x \geq 2$, we have
   $$\sum_{n \leq x} \tau^2(n) \ll x \log^3 x.$$

   2). For $x \geq 2$, we have
   $$\sum_{n \leq x} \frac{\tau(n)}{n} = \frac{1}{2} \log^2 x + 2\gamma \log x + O(1),$$
   where γ is the Euler constant.

   3). Let $k \geq 1$ be a positive integer. Then for $x \geq 2$, we have
   $$\sum_{n \leq x} \frac{\tau^k(n)}{n} \ll \log^{2^k} x \quad \text{and} \quad \sum_{n \leq x} \tau^k(n) \ll x \log^{2^k - 1},$$
   where the implied constants may depend on k.

2. Prove that
   $$M(x) := \sum_{n \leq x} \mu(n) = o(x), \quad (x \to +\infty)$$
   is equivalent to
   $$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0.$$
   Hint: Consider the identity
   $$\sum_{n \leq x} \mu(n) \left[\frac{x}{n}\right] = 1.$$

3. The goal of this exercise is to give a lower bound for $\varphi(n)$.

   1). Let $\omega(n)$ denote the number of distinct prime factors of n. Show that for $n \geq 2$,
   $$\omega(n) \ll \log n.$$

   2). Let p_n denote the n-th prime number. Show that
   $$p_n \asymp n \log n.$$

   3). Show that for $n \geq 30$,
   $$\varphi(n) \gg \frac{n}{\log \log n}.$$

   Hint: Apply the identity
   $$\varphi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right)$$
   and Mertens' theorem.

4. Prove Selberg's identity:
$$\psi(x) \log x + \sum_{n \leq x} \Lambda(n) \psi\left(\frac{x}{n}\right) = 2x \log x + O(x).$$