Chapter 6

The analytic continuation of $\zeta(s)$

Throughout this chapter. The letter s denotes a complex variable. The real numbers σ and t are implicitly defined by $s = \sigma + it$.

6.1 Analytic continuation of $\zeta(s)$ in $\operatorname{Re} s > 0$

Theorem 6.1. *For* $\operatorname{Re} s > 1$*, we have*

$$\zeta(s) = \left(2^{1-s} - 1\right)^{-1} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}. \tag{6.1}$$

The series in the right-side of (6.1) *is uniformly convergent in any compact subset of* $\operatorname{Re} s > 0$*. As a consequence,* (6.1) *gives the analytic continuation of* $\zeta(s)$ *to the half-plane* $\operatorname{Re} s > 0$*.*

Proof. For $\operatorname{Re} s > 1$, we have

$$2^{-s}\zeta(s) = \frac{1}{2^s} \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = \sum_{\substack{n=1 \\ n \text{ is even}}} \frac{1}{n^s}.$$

Therefore,

$$(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} - 2 \sum_{\substack{n=1 \\ n \text{ is even}}} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}.$$

This gives (6.1).

Now we fix a compact subset D of $\operatorname{Re} s > 0$. For $s = \sigma + it \in D$, we have $\sigma > \sigma_0$ for some $\sigma_0 > 0$ and $|s| \le C$ for some constant C. For integers $N < M$, we have

$$\begin{aligned}\left|\sum_{N\le n\le M} \frac{(-1)^n}{n^s}\right| &\le \frac{1}{N^\sigma} + \sum_{N\le n\le M}\left|\frac{1}{n^s} - \frac{1}{(n+1)^s}\right| \\ &\le \frac{1}{N^\sigma} + \sum_{n\le n\le M}\left|s\int_n^{n+1}\frac{\mathrm{d}x}{x^{s+1}}\right| \\ &\ll \frac{1}{N^\sigma} + \int_N^{M+1}\frac{\mathrm{d}x}{x^{\sigma+1}} \\ &\ll \frac{1}{N^\sigma} + \frac{1}{\sigma}\frac{1}{N^\sigma} \\ &\ll \frac{1}{N^{\sigma_0}}\end{aligned}$$

where the implied constant depends on D. Thus the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s}$$

is uniformly convergent in D hence defines a holomorphic function. □

Corollary 6.2. *The Riemann zeta function $\zeta(s)$ has a simple pole at $s = 1$ with residue 1, and $\zeta(s)$ is negative on the segment $0 < \sigma < 1$, $t = 0$.*

Proof. For $s \to 1$, we have

$$2^{1-s} - 1 = (\log 2)(1-s) + O\left(|s-1|^2\right).$$

Thus as $s \to 1$,

$$(s-1)\zeta(s) = \frac{s-1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s} = \frac{\log 2}{\log 2 + O\left(|s-1|\right)} \to 1$$

where we haved used the fact that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \log 2.$$

This implies that $s = 1$ is a simple pole of $\zeta(s)$ with residue 1.

For the second assertion, we need only to notice that for $0 < \sigma < 1$, we have

$$2^{1-\sigma} - 1 > 0 \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{(-1)^n}{n^\sigma} < 0.$$

□

6.2 Analytic continuation of $\zeta(s)$ in $\operatorname{Re} s > -1$

By the Euler–Maclaurin formula, we can give the analytic continuation of $\zeta(s)$ on $\operatorname{Re} s > 0$ in another way. For $\operatorname{Re} s > 1$ (in fact, we can assume $\operatorname{Re} s$ is sufficiently large, e.g. $\operatorname{Re} s > 100$), by the Euler–Maclaurin summation formula (Theorem 3.3), we have

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \sum_{n>1} \frac{1}{n^s} \\ &= 1 + \int_1^{\infty} \frac{\mathrm{d}x}{x^s} - \frac{1}{2} + s\int_1^{\infty} \frac{\rho(x)}{x^{s+1}}\,\mathrm{d}x \\ &= \frac{1}{2} + \frac{1}{s-1} + s\int_1^{\infty} \frac{\rho(x)}{x^{s+1}}\,\mathrm{d}x.\end{aligned} \tag{6.2}$$

The last integral is absolutely and uniformly convergent in any compact subset of $\operatorname{Re} s > 0$, so we regain the analytic continuation of $\zeta(s)$ in $\operatorname{Re} s > 0$. Furthermore, we can immediately see that $s = 1$ is a simple pole of $\zeta(s)$ with residue 1. Write

$$\sigma(x) = \int_0^x \rho(t)\,\mathrm{d}t.$$

We again use the partial integral for the last integration, getting

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + s(s+1)\int_1^{\infty} \frac{\sigma(x)}{x^{s+2}}\,\mathrm{d}x. \tag{6.3}$$

The last integral is absolutely convergent for $\operatorname{Re} s > -1$. So (6.3) gives the analytic continuation of $\zeta(s)$ in $\operatorname{Re} s > -1$. Actually, we can repeat the above process to give the analytic continuation of $\zeta(s)$ in any right half-plane.

Corollary 6.3. *We have $\zeta(0) = -1/2$.*

Proof. Take $s = 0$ in (6.3). □

Corollary 6.4. *We have*

$$\lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right) = \gamma,$$

where γ is the Euler constant defined by

$$\gamma = \lim_{N\to\infty}\left(\sum_{n\le N} \frac{1}{n} - \log N\right).$$

Therefore, as $s \to 1$, we have

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|).$$

Proof. By (6.2), we have

$$\begin{aligned}
\lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right) &= \frac{1}{2} + \int_1^\infty \frac{\rho(x)}{x^2}\,\mathrm{d}x \\
&= \frac{1}{2} + \lim_{N\to\infty}\int_1^N \frac{\frac{1}{2} - x + [x]}{x^2}\,\mathrm{d}x \\
&= \frac{1}{2} + \lim_{N\to\infty}\left(\frac{1}{2} - \frac{1}{2N} - \log N + \sum_{n=1}^{N-1} n\int_n^{n+1}\frac{\mathrm{d}x}{x^2}\right) \\
&= 1 + \lim_{N\to\infty}\left(\sum_{n=1}^{N-1} n\left(\frac{1}{n} - \frac{1}{n+1}\right) - \log N\right) \\
&= 1 + \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n} - \frac{N-1}{N} - \frac{1}{N} - \log N\right) \\
&= \lim_{N\to\infty}\left(\sum_{n\le N}\frac{1}{n} - \log N\right) \\
&= \gamma.
\end{aligned}$$

□

6.3 Estimate of $\zeta(s)$ in the critical strip

If we start summing from $n = N$, we can get a variant of (6.3):

$$\begin{aligned}
\zeta(s) &= \sum_{n\le N}\frac{1}{n^s} + \int_N^\infty \frac{\mathrm{d}x}{x^s} + \frac{\rho(x)}{x^s}\bigg|_N^\infty + s\int_N^\infty \frac{\rho(x)}{x^{s+1}}\,\mathrm{d}x \\
&= \sum_{n\le N}\frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} + s(s+1)\int_N^\infty \frac{\sigma(x)}{x^{s+2}}\,\mathrm{d}x.
\end{aligned} \tag{6.4}$$

The advantage of this formula is that the parameter N can be freely selected. Trivially estimate the last integral, we obtain the following estimate for $\zeta(s)$:

Theorem 6.5. *For $\sigma > -1$ and $N \geq 1$, we have*

$$\zeta(s) = \sum_{n\leq N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} + O\left(\frac{|s(s+1)|}{\sigma+1}N^{-\sigma-1}\right),$$

where the implied constant is absolute.

In practice, the growth of $\zeta(s)$ as $|t| \to \infty$ is critical. However, the above estimate can not provide a satisfactory result. The following formula, which can be considered as the prototype of the so-called approximate functional equation, gives a better estimate.

Theorem 6.6. *For $s = \sigma + it$ with $\sigma \geq 0$, $|t| \leq 2T$ and $T \geq 1$ we have*

$$\zeta(s) = \sum_{n\leq T} n^{-s} + \frac{T^{1-s}}{s-1} + O(T^{-\sigma}),$$

where the implied constant is absolute.

To prove Theorem 6.6, we need a result on exponential sums.

Lemma 6.7. *Let $f(x)$ be a real function with $|f'(x)| \leq 1-\theta$ and $f''(x) \neq 0$ on $[a, b]$. We then have*

$$\sum_{a<n\leq b} g(n)e(f(n)) = \int_a^b g(x)e(f(x))\,\mathrm{d}x + O(G\theta^{-1})$$

where

$$G = |g(b)| + \int_a^b |g'(t)|\,\mathrm{d}t.$$

Proof. This is Lemma 8.8 in *Analytic Number Theory* by Iwaniec and Kowalski. □

Proof of Theorem 6.6. Taking $N = [T^2]$ in Theorem 6.5, we obtain that

$$\begin{aligned}\zeta(s) &= \sum_{n\leq N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} + O\left(\frac{|s(s+1)|}{\sigma+1}N^{-\sigma-1}\right) \\ &= \sum_{n\leq N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + O(T^{-\sigma}) \qquad (6.5) \\ &= \sum_{n\leq T} \frac{1}{n^s} + \sum_{T<n\leq N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} + O(T^{-\sigma}).\end{aligned}$$

Now we use Lemma 6.7 to deal with the sum

$$\sum_{T<n\le N} \frac{1}{n^s}.$$

Apply Lemma 6.7 for

$$g(x) = \frac{1}{x^\sigma} \quad \text{and} \quad f(x) = -\frac{t}{2\pi}\log x.$$

Then we have

$$G \ll T^{-\sigma} \quad \text{and} \quad |f'(x)| \le \frac{1}{\pi}$$

for $x \in [T, N]$. Notice that

$$\frac{1}{x^s} = g(x)e(f(x)).$$

So by Lemma 6.7, we have

$$\sum_{T<n\le N} \frac{1}{n^s} = \int_T^N \frac{1}{x^s}\,\mathrm{d}x + O(T^{-\sigma}) = \frac{T^{1-s} - N^{1-s}}{1-s} + O(T^{-\sigma}). \tag{6.6}$$

Substituting (6.6) into (6.5), we complete the proof. □

Corollary 6.8. *Let $T \ge 2$ and let $s = \sigma + it$ be such that*

$$1 - \frac{1}{\log T} \le \sigma \le 2, \qquad |t| \le T.$$

Then we have

$$\zeta(s) - \frac{1}{s-1} = O\left(\log T\right)$$

and

$$\zeta'(s) + \frac{1}{(s-1)^2} = O(\log^2 T).$$

Proof. By Theorem 6.6, we have

$$\zeta(s) = \sum_{n\le T} n^{-s} + \frac{T^{1-s}-1}{s-1} + O(T^{-\sigma}).$$

We estimate each terms. One has

$$\frac{T^{1-s}-1}{s-1} = \int_1^T \frac{\mathrm{d}x}{x^s}.$$

By the Euler–Macluarion formula, we have

$$\sum_{n\le T} n^{-s} = \int_1^T \frac{\mathrm{d}x}{x^s} + O(1) + O\left(|s|\int_1^T \frac{\mathrm{d}x}{x^{\sigma+1}}\right) = \int_1^T \frac{\mathrm{d}x}{x^s} + O(T^{1-\sigma}).$$

Since $\sigma > 1 - 1/\log T$, the O-term is

$$\ll T^{1/\log T} \ll 1.$$

So it remains to bound

$$\int_1^T \frac{\mathrm{d}x}{x^s}.$$

We have

$$\left|\int_1^T \frac{\mathrm{d}x}{x^s}\right| \le \int_1^T \frac{\mathrm{d}x}{x^\sigma} \le \int_1^T \frac{\mathrm{d}x}{x^{1-1/\log T}} \ll \log T.$$

This completes the proof of the first assertion.

For the second assertion, we first notice that the constants are not necessary in the condition

$$1 - \frac{1}{\log T} \le \sigma \le 2, \qquad |t| \le T.$$

We can replace this condition by

$$1 - \frac{C_1}{\log T} \le \sigma \le C_2, \qquad |t| \le T.$$

with any positive constant C_1 and $C_2 > 1$. Then the conclusion is still valid (of course, the implied constant in the error term may depend on C). By differentiating (6.2), we see that

$$\zeta'(s) + \frac{1}{(s-1)^2}$$

is holomorphic in $\operatorname{Re} s > 0$. So for $|t| \le 3$, we have

$$\zeta'(s) + \frac{1}{(s-1)^2} = O(1).$$

Thus it suffices to prove

$$\zeta'(s) = O(\log^2 T).$$

for

$$1 - \frac{1}{\log T} \le \sigma \le 2, \qquad 3 \le |t| \le T.$$

By Cauchy's formula, we have

$$\zeta'(s) = \frac{1}{2\pi i} \int_{|z-s|=r} \frac{\zeta(z)}{(z-s)^2} \, \mathrm{d}z$$

for any $r > 0$. We specify $r = 1/\log T$. Then the variable z in the integral satisfies

$$1 - \frac{2}{\log T} \leq \operatorname{Re} z \leq 3, \qquad 2 \leq |\operatorname{Im} z| \leq T + 1.$$

So by the first assertion, we have

$$\zeta(s) = \frac{1}{s-1} + O(\log T) = O(\log T).$$

Thus

$$|\zeta'(s)| \ll \int_{|z-s|=r} \frac{|\zeta(z)|}{|z-s|^2} \, \mathrm{d}z \ll \frac{1}{\log T} \cdot \frac{\log T}{1/\log^2 T} = \log^2 T.$$

□