Chapter 5

The Dirichlet series and Perron's formula

Throughout this chapter. The letter s denotes a complex variable. The real numbers σ and t are implicitly defined by $s = \sigma + it$.

5.1 Convergence of Dirichlet series

Recall the definition of Dirichlet series:

Definition 5.1 (Dirichlet Series)**.** *Let f be an arithmetic functions. The Dirichlet series associated with f is the function $D(f;s)$ of the complex variable s defined by*

$$D(f;s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \tag{5.1}$$

for those s where the series (5.1) *is convergent.*

Theorem 5.1. *Let $D(f;s)$ be a Dirichlet series and let $\sigma_0 \in \mathbb{R}$. If $D(f;s_0)$ is absolutely convergent for some $s_0 \in \mathbb{C}$ with $\operatorname{Re} s_0 = \sigma_0$, then it converges absolutely and uniformly in the closed half-plane*

$$\{s \in \mathbb{C} : \operatorname{Re} s \geq \sigma_0\}.$$

Proof. Since for any s with $\operatorname{Re} s \geq \sigma_0$,

$$\left|\frac{f(n)}{n^s}\right| \leq \left|\frac{f(n)}{n^{\sigma_0}}\right| = \left|\frac{f(n)}{n^{s_0}}\right|,$$

the desired conclusion follows by the Weierstrass criterion. □

Definition 5.2 (Abscissa of absolute convergence). *Let $D(f;s)$ be a Dirichlet series. Define the* ***abscissa of absolute convergence*** *of $D(f;s)$ by*

$$\sigma_a = \inf\left\{\sigma \in \mathbb{R} \,:\, D(f;\sigma) \text{ is absolutely convergent}\right\}.$$

That is to say, $D(f;s)$ is absolutely convergent for $\operatorname{Re} s > \sigma_a$ and is not absolutely convergent for $\operatorname{Re} s < \sigma_a$. By convention we allow $\sigma_a = \pm\infty$.

Example 5.1. *The abscissa of absolute convergence of $\zeta(s) = D(\mathbb{1};s)$ is $\sigma_a = 1$. By Proposition 3.1, the σ_a of $D(\tau;s)$ is 1.*

In Chapter 2, we have discussed the relationship between Dirichlet series and Dirichlet convolutions. In practice, we need to consider the derivative of a Dirichlet series. So we need the following convergence theorems.

Theorem 5.2. *Suppose that f, g and h are arithmetic functions such that $h = f * g$. If both $D(f;s)$ and $D(g;s)$ are absolutely convergent at any given point s, then so is $D(h;s)$. Moreover, we have*

$$D(h;s) = D(f;s)D(g;s).$$

Proof. For any $x \geq 1$, we have

$$\sum_{n\leq x}\left|\frac{h(n)}{n^s}\right| = \sum_{md\leq x}\left|\frac{f(m)g(d)}{(md)^s}\right| \leq \sum_{m\leq x}\left|\frac{f(m)}{m^s}\right|\sum_{d\leq x}\left|\frac{g(d)}{d^s}\right|.$$

Let $x \to +\infty$ and the desired conclusion follows. □

Corollary 5.3. *Let f be a completely multiplicative function. Then $D(f;s) \neq 0$ in $\operatorname{Re} s > \sigma_a$ and*

$$\frac{1}{D(f;s)} = \sum_{n=1}^{\infty}\frac{\mu(n)f(n)}{n^s}.$$

In particular, the Riemann zeta function $\zeta(s) \neq 0$ in the half-plane $\operatorname{Re} s > 1$.

Proof. By Theorem 2.5, we have

$$f^{-1}(n) = \mu(n)f(n)$$

where $f^{-1}(n)$ denote the Dirichlet inverse of $f(n)$. Moreover, since $|\mu(n)| \leq 1$, the series

$$\sum_{n=1}^{\infty}\frac{\mu(n)f(n)}{n^s}$$

is also absolutely and uniformly convergent in any closed half-plane in $\operatorname{Re} s > \sigma_a$. The desired result follows from Theorem 5.2. □

Theorem 5.4. *Suppose that $D(f;s)$ is a Dirichlet series with the abscissa of absolute convergence σ_a. Then for any $s \in \mathbb{C}$ with $\operatorname{Re} s > \sigma_a$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}s} D(f;s) = -\sum_{n=1}^{\infty} \frac{f(n)\log n}{n^s}.$$

That is, we can termwise calculate the derivative of $D(f;s)$.

Proof. Since $\log n \ll_\varepsilon n^\varepsilon$ for any $\varepsilon > 0$, we see that

$$\sum_{n=1}^{\infty} \frac{f(n)\log n}{n^s}$$

is absolutely and uniformly convergent in any closed half-plane in $\operatorname{Re} s > \sigma_a$. The desired result follows from the Weierstrass criterion. □

Remark. In fact, since $f(n)/n^s$ is holomorphic as a function in s, we can directly get the conclusion of Theorem 5.4 by complex analysis.

Example 5.2. *For $\operatorname{Re} s > 1$, we have*

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

and

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

Moreover, these two Dirichlet series are absolutely convergent in $\operatorname{Re} s > 1$.

5.2 Dirichlet series associated with multiplicative functions

Formally, if f is a multiplicative function, by the fundamental theorem of arithmetic, we can write

$$D(f;s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots\right).$$

We give a rigorous proof of this fact.

Lemma 5.5. *If*

$$\sum_{n=1}^{\infty} |a_n| < \infty,$$

then the product

$$\prod_{n=1}^{\infty} (1 + a_n)$$

converges. Moreover, the product converges to 0 *if and only if one of its factors is* 0.

Proof. Since $\sum_{n=1}^{\infty} |a_n|$ is convergent, we have $|a_n| < 1/2$ for sufficiently large n. Without loss of generality, we can assume this property holds for all n. We define $\log z$ by the usual power series, and this logarithm satisfies the property that

$$1 + z = \exp\left(\log(1 + z)\right), \quad |z| < 1.$$

Then we have

$$\prod_{n=1}^{N} (1 + a_n) = \exp\left(\sum_{n=1}^{N} \log(1 + a_n)\right).$$

So it suffices to prove that

$$\sum_{n=1}^{N} \log(1 + a_n)$$

is convergent. By the power series expansion, we see that

$$|\log(1 + z)| \le 2|z|$$

for $|z| < 1/2$. Since $\sum_{n=1}^{\infty} |a_n|$ is convergent, we obtain the desired conclusion. □

Theorem 5.6. *Let f be a multiplicative function and let s be a complex number. If $D(f; s)$ is absolutely convergent, then*

$$D(f; s) = \prod_{p} \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}}. \tag{5.2}$$

Furthermore, if f is completely multiplicative, the equation (5.2) *takes the simpler form*

$$D(f; s) = \prod_{p} \left(1 - \frac{f(p)}{p^s}\right)^{-1}.$$

Proof. Since $D(f;s)$ is absolutely convergent and

$$\prod_p \sum_{k=1}^{\infty} \left| \frac{f(p^k)}{p^{ks}} \right| \le \sum_{n=1}^{\infty} \left| \frac{f(n)}{n^s} \right|,$$

we see from Lemma 5.5 that the product

$$\prod_p \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} = \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{ks}} \right)$$

is convergent. Let $P^+(n)$ denote the largest prime factor of n. The inequality

$$\left| \sum_{n=1}^{\infty} \frac{f(n)}{n^s} - \prod_{p \le x} \sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} \right| = \left| \sum_{P^+(n) > x} \frac{f(n)}{n^s} \right| \le \sum_{n > x} \left| \frac{f(n)}{n^s} \right|$$

implies (5.2) by letting $x \to +\infty$.

If f is completely multiplicative, then we have

$$\frac{f(p^k)}{p^{ks}} = \left(\frac{f(p)}{p^s} \right)^k.$$

So we have

$$\sum_{k=0}^{\infty} \frac{f(p^k)}{p^{ks}} = \left(1 - \frac{f(p)}{p^s} \right)^{-1}.$$

Note that since $D(f;s)$ is absolutely convergent, we must have $|f(p)/p^s| < 1$. $\square$

Corollary 5.7 (Euler product)**.** *For* $\operatorname{Re} s > 1$*, we have the following Euler product formula for the Riemann zeta function:*

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p} \right)^{-1}.$$

5.3 Perron's formula

Let $f(s)$ be a funtion in complex variable and let $c \in \mathbb{R}$. Denote

$$\int_{(c)} f(s) \, \mathrm{d}s = \lim_{T \to +\infty} \int_{c-iT}^{c+iT} f(s) \, \mathrm{d}s.$$

Lemma 5.8. *For any $c > 0$, we have*

$$\frac{1}{2\pi i}\int_{(c)} \frac{x^s}{s}\,\mathrm{d}s = \begin{cases} 1, & x > 1, \\ 0, & 0 < x < 1, \\ 1/2, & x = 1. \end{cases}$$

We omit the proof since we will soon prove a quantitative version of this lemma.

Theorem 5.9 (Perron's formula)**.** *Let $D(f;s)$ be a Dirichlet series. Then for any $c > \sigma_a$ and any non-integer $x > 1$, we have*

$$\sum_{n\le x} f(n) = \frac{1}{2\pi i}\int_{(c)} D(f;s)\frac{x^s}{s}\,\mathrm{d}s.$$

"Proof". Exchanging the integral and the sum, we obtain that

$$\frac{1}{2\pi i}\int_{(c)} D(f;s)\frac{x^s}{s}\,\mathrm{d}s = \frac{1}{2\pi i}\int_{(c)}\sum_{n=1}^{\infty}\frac{f(n)}{n^s}\frac{x^s}{s}\,\mathrm{d}s = \sum_{n=1}^{\infty} f(n)\left(\frac{1}{2\pi i}\int_{(c)}\left(\frac{x}{n}\right)^s\frac{1}{s}\,\mathrm{d}s\right).$$

By Theorem 5.8, the integral in the last sum is

$$\frac{1}{2\pi i}\int_{(c)}\left(\frac{x}{n}\right)^s\frac{1}{s}\,\mathrm{d}s = \begin{cases} 1, & n < x, \\ 0, & n > x. \end{cases}$$

Note that since $x \notin \mathbb{Z}$, $n = x$ could not happen. The desired conclusion follows. □

One may notice that the first step is not rigorous in our "proof". We have two ways to overcome this problem. One is to use the Laplace transform, which is left as an exercise. The other is to truncate the integral and obtain an estimate of the form

$$\left|\sum_{n\le x} f(n) - \frac{1}{2\pi i}\int_{c-iT}^{c+iT} D(f;s)\frac{x^s}{s}\,\mathrm{d}s\right| = o_{T\to+\infty}(1)\,. \tag{5.3}$$

Letting $T \to +\infty$, we obtain Perron's formula. However, in the application of Perron's formula, we need to truncate the integral. So anyway, we need to give an effective version of Perron's formula.

5.4 Effective Perron formula

Lemma 5.10. *Define the function $\delta(x)$ by (not the arithmetic function δ defined in Chapter 2)*

$$\delta(x) = \begin{cases} 1, & x > 1, \\ 1/2, & x = 1, \\ 0, & 0 < x < 1. \end{cases}$$

Then for any $c > 0$, $T \geq 2$ and $x > 0$, we have

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s} \, \mathrm{d}s = \delta(x) + \begin{cases} O\left(x^c \min\left(1, \frac{1}{T|\log x|}\right)\right), & x \neq 1, \\ O\left(\frac{1}{T}\right), & x = 1. \end{cases}$$

Proof. We first assume $0 < x < 1$. We move the integration to the horizontal segment $s = \alpha \pm iT$ with $c < \alpha < \infty$ getting

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s} \, \mathrm{d}s \ll \frac{1}{T} \int_c^\infty x^\sigma \, \mathrm{d}\sigma \ll \frac{x^c}{T|\log x|}.$$

If we move the integration to the right arc of the circle $|s| = |c + iT|$ then we get

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s} \, \mathrm{d}s \ll x^c.$$

So we prove the case $0 < x < 1$.

For $x > 1$, the proof is similar. But we should move the integration to the left horizontal segment and the left arc of the circle. The pole at $s = 0$ contributes the residue 1.

Finally, for $x = 1$, we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{s} \, \mathrm{d}s &= \frac{1}{2\pi i} \left(\log(c + iT) - \log(c - iT)\right) \\ &= \frac{1}{\pi} \arctan \frac{T}{\sqrt{c^2 + T^2}} \\ &= \frac{1}{2} + O\left(\frac{1}{T}\right). \end{aligned}$$

□

Theorem 5.11 (Effective Perron formula)**.** *Let $D(f;s)$ be a Dirichlet series with σ_a the abscissa of absolutely convergence. For $x \geq 2$ and $\sigma > \sigma_a$, define*

$$A(x) = \max_{x/2 \leq n \leq 3x/2} |f(n)|, \quad B(\sigma) = \sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma}}.$$

Then for $T \geq 2$, $c > \sigma_a$ and $x \geq 2$ with $x = N + 1/2$ for some $N \in \mathbb{Z}$, we have

$$\sum_{n \leq x} f(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} D(f;s) \frac{x^s}{s} ds + O\left(\frac{xA(x)\log x}{T}\right) + O\left(\frac{x^c B(c)}{T}\right).$$

Proof. By Lemma 5.10, we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} D(f;s) \frac{x^s}{s} ds &= \sum_{n=1}^{\infty} f(n) \left(\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(\frac{x}{n}\right)^s \frac{\mathrm{d}s}{s}\right) \\ &= \sum_{n \leq x} f(n) + O(R) \end{aligned}$$

where

$$R = x^c \sum_{n=1}^{\infty} \frac{|f(n)|}{Tn^c |\log(x/n)|}.$$

We divide the sum into three parts:

$$R = x^c \left\{ \sum_{n < x/2} + \sum_{x/2 \leq n \leq 3x/2} + \sum_{n > 3x/2} \right\} \frac{|f(n)|}{Tn^c |\log(x/n)|}.$$

For $n < x/2$ or $n > 3x/2$, we have

$$\left|\log \frac{x}{n}\right| \geq \log 2 \gg 1.$$

So the first sum and the third sum are bounded by

$$\frac{x^c}{T} \left\{ \sum_{n < x/2} + \sum_{n > 3x/2} \right\} \frac{|f(n)|}{n^c} \ll \frac{x^c B(c)}{T}.$$

For $x/2 \leq n \leq 3x/2$, $1/|\log(x/n)|$ could be large. So we should consider its contribution. Recall that $x = N + 1/2$. We divide the sum into

$$\sum_{x/2 \leq n \leq N} + \sum_{N+1 \leq n \leq 3x/2}.$$

The treatment of these two sums are the same, so we only deal with the latter one. For $N+1 \leq n \leq 3x/2$, we have

$$\left|\log \frac{x}{n}\right| = \log \frac{n}{x} = \log\left(1 + \frac{x-n}{x}\right) \gg \frac{n-x}{x}.$$

So

$$x^c \sum_{N+1 \leq n \leq 3x/2} \frac{|f(n)|}{Tn^c\,|\log(x/n)|} \ll \frac{xA(x)}{T} \sum_{N+1 \leq n \leq 3x/2} \frac{1}{n-x} \ll \frac{xA(x)\log x}{T}.$$

The same estimate holds for

$$x^c \sum_{x/2 \leq n \leq N} \frac{|f(n)|}{Tn^c\,|\log(x/n)|}.$$

Combining all the above, we get the desired conclusion. □

5.5 The expression of $\psi(x)$

Now we would like to deduce a integral expression of $\psi(x)$ via Perron's formula. First we need to know the behaviour of $\zeta(s)$ near $s=1$.

Lemma 5.12. *For $k \geq 0$ and $\operatorname{Re} s > 1$, we have*

$$\zeta^{(k)}(s) = \frac{(-1)^k k!}{(s-1)^{k+1}} + O((\log 2|s|)^{k+1})$$

where the implies constant depends only on k.

Proof. Let $\sigma = \operatorname{Re} s$. Let $X \geq 2$ be a large parameter specified later. By the

Euler–Maclaurin summation formula (Theorem 3.3), we have

$$\begin{aligned}
(-1)^k\zeta^{(k)}(s) &= \sum_{n=1}^{\infty}\frac{(\log n)^k}{n^s} \\
&= \sum_{n\le X}\frac{(\log n)^k}{n^s} + \int_X^{+\infty}\frac{(\log u)^k}{u^s}\,\mathrm{d}s - \frac{\rho(X)(\log X)^k}{X^s} \\
&\quad - \int_X^{+\infty}\rho(u)\frac{k(\log u)^{k-1}u^{s-1} - su^{s-1}(\log u)^k}{u^{2s}}\,\mathrm{d}u \\
&= \sum_{n\le X}\frac{(\log n)^k}{n^s} + \int_X^{+\infty}\frac{(\log u)^k}{u^s}\,\mathrm{d}s + O\left(\frac{(\log X)^k}{X^\sigma}\right) \\
&\quad + O\left(\int_X^{+\infty}\frac{k(\log u)^{k-1} + |s|(\log u)^k}{u^{\sigma+1}}\,\mathrm{d}u\right) \\
&= \sum_{n\le X}\frac{(\log n)^k}{n^s} + \int_X^{+\infty}\frac{(\log u)^k}{u^s}\,\mathrm{d}s + O\left(|s|\int_X^{+\infty}\frac{(\log u)^k}{u^{\sigma+1}}\,\mathrm{d}u\right) \\
&= \int_1^{+\infty}\frac{(\log u)^k}{u^s}\,\mathrm{d}s + \sum_{n\le X}\frac{(\log n)^k}{n^s} - \int_1^{X}\frac{(\log u)^k}{u^s}\,\mathrm{d}s \\
&\quad + O\left(|s|\int_X^{+\infty}\frac{(\log u)^k}{u^{\sigma+1}}\,\mathrm{d}u\right).
\end{aligned}$$

By repeatedly using partial integration, we obtain that

$$\int_1^{+\infty}\frac{(\log u)^k}{u^s}\,\mathrm{d}s = \frac{k}{s-1}\int_1^{+\infty}\frac{(\log u)^{k-1}}{u^s}\,\mathrm{d}s = \cdots = \frac{k!}{(s-1)^k}\int_1^{+\infty}\frac{\mathrm{d}s}{u^s} = \frac{k!}{(s-1)^{k+1}}.$$

It is the desired main term. Since $\sigma = \operatorname{Re} s > 1$, we have

$$\sum_{n\le X}\frac{(\log n)^k}{n^s} \ll (\log X)^k\sum_{n\le X}\frac{1}{n} \ll (\log X)^{k+1}.$$

Clearly, the same estimate holds for

$$\int_1^{X}\frac{(\log u)^k}{u^s}\,\mathrm{d}s.$$

Finally, we have

$$\begin{aligned}\int_X^{+\infty} \frac{(\log u)^k}{u^{\sigma+1}}\,\mathrm{d}u &= -\frac{1}{\sigma}\int_X^{+\infty} (\log u)^k \,\mathrm{d}u^{-\sigma} \\ &= \frac{(\log X)^k}{\sigma X^\sigma} + \frac{1}{\sigma}\int_X^{+\infty} \frac{(\log u)^{k-1}}{u^{\sigma+1}}\,\mathrm{d}u \\ &\ll \frac{(\log X)^k}{X} + \int_X^{+\infty} \frac{(\log u)^{k-1}}{u^{\sigma+1}}\,\mathrm{d}u \\ &\cdots \\ &\ll \frac{(\log X)^k}{X}.\end{aligned}$$

As a summary of the above argument, we obtain that

$$\zeta^{(k)}(s) = \frac{(-1)^k k!}{(s-1)^{k+1}} + O\left(\left(1+\frac{|s|}{X}\right)(\log X)^{k+1}\right).$$

Taking $X = 2|s|$, we obtain the desired conclusion. □

Corollary 5.13. *For* $1 < \sigma < 2$*, we have*

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} = \frac{1}{\sigma - 1} + O(1).$$

Proof. By Lemma 5.12, for $1 < \sigma < 2$, we have

$$\zeta'(\sigma) = -\frac{1}{(\sigma-1)^2} + O(1), \quad \zeta(\sigma) = \frac{1}{\sigma-1} + O(1).$$

Therefore, we have

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} = \frac{1}{\sigma - 1} + O(1).$$

□

Theorem 5.14. *Let* $x \geq 2$ *be such that* $x = N + 1/2$ *for some* $N \in \mathbb{Z}$*. Then for* $T \geq 2$ *and* $c = 1 + 1/\log x$*, we have*

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}\,\mathrm{d}s + O\left(\frac{x\log^2 x}{T}\right).$$

Proof. We apply Theorem 5.11 for $-\zeta'(s)/\zeta(s)$. Let $A(x)$ and $B(\sigma)$ be as in Theorem 5.11. We have

$$A(x) = \max_{x/2 \le n \le 3x/2} |\Lambda(n)| \ll \log x.$$

By Corollary 5.13, we have

$$B(c) = B(1 + 1/\log x) = \log x + O(1) \ll \log x.$$

So we have

$$\frac{xA(x)\log x}{T} \ll \frac{x \log^2 x}{T}$$

and

$$\frac{x^c B(c)}{T} = \frac{exB(c)}{T} \ll \frac{x \log x}{T}.$$

By Theorem 5.11, we obtain the desired result. □