## §1 The gamma function

For $\operatorname{Re} s > 0$, the gamma function is defined by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} \, dt.$$

This integral is absolutely convergent because near $t=0$, the function $t^{\operatorname{Re} s - 1}$ is integrable and for $t$ large, $e^{-t}$ is rapidly decreasing.

**Lemma 9.1** For $\operatorname{Re} s > 0$, we have

$$\Gamma(s+1) = s\Gamma(s).$$

Thus $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \cdots$

*Proof.* Integrating by parts gives

$$\Gamma(s+1) = \int_0^{+\infty} e^{-t} t^s \, dt = -e^{-t} t^s \Big|_0^{+\infty} + s\int_0^{+\infty} e^{-t} t^{s-1} \, dt = s\Gamma(s).$$

Now it suffices to check $\Gamma(1) = 1$, which is obvious. $\square$

$$\Gamma(1) = \int_0^{+\infty} e^{-t} \, dt = 1.$$

**Theorem 9.2** The function $\Gamma(s)$ has an analytic continuation to a meromorphic function on $\mathbb{C}$ whose only singularities are simple poles at the negative integers $s = 0, -1, \cdots$. The residue of $\Gamma$ at $s = -n$ is $(-1)^n / n!$

*Proof.* For $\operatorname{Re} s > -1$, define

$$F_1(s) = \frac{\Gamma(s+1)}{s}$$

Then $F_1$ is meromorphic in $\mathrm{Re}\, s > -1$ of which the only singularity is a simple pole at $s=0$. The residue is

$$\mathrm{Res}_{s=0} F_1(s) = \lim_{s\to 0} \Gamma(s+1) = \Gamma(1) = 1 .$$

Moreover, for $\mathrm{Re}\, s > 0$, we have $F_1(s) = \Gamma(s)$ by Lema 9.1. So $F_1$ is the analytic continuation of $\Gamma(s)$ in $\mathrm{Re}\, s > -1$. Similarly, for any $m \in \mathbb{N}^*$, we can extend $\Gamma(s)$ to the half-plane $\mathrm{Re}\, s > -m$ by taking

$$F_m(s) = \frac{\Gamma(s+m)}{s(s+1)\cdots(s+m-1)}$$

To evaluate the residue at $s=-n$, we need only to consider

$$F_{n+1}(s) = \frac{\Gamma(s+n+1)}{s(s+1)\cdots(s+n)}$$

and calculate that

$$\mathrm{Res}_{s=-n} F_{n+1}(s) = \lim_{s\to -n} \frac{\Gamma(s+n+1)}{s(s+1)\cdots(s+n-1)} = \frac{1}{(-n)(-n+1)\cdots(-1)}$$

$$= \frac{(-1)^n}{n!} .$$

$\square$

## §9.2 The functional equation

Theorem 9.3

Poisson summation formula. Let $f$ be a rapidly decreasing function (i.e. $|f^{(k)}(x)| \ll_{A,k} |x|^{-k}$). Let $\hat{f}(x)$ be the Fourier transform of $f(x)$ defined by

$$\hat{f}(x) = \int_{\mathbb{R}} f(t)\, e^{-2\pi i x t}\, dt$$

Then we have

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

Proof. Let $g(x) = \sum_{n\in\mathbb{Z}} f(x+n)$. Then $g(x)$ is a continuous function with period 1. So the Fourier series for $g(x)$ converges to $g(x)$.

Let

$$g(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}.$$

Then we have

$$c_n = \int_0^1 g(x) e^{-2\pi i n x} dx = \sum_{m\in\mathbb{Z}} \int_0^1 f(m+x) e^{-2\pi i n x} dx$$

$$= \sum_{m\in\mathbb{Z}} \int_m^{m+1} f(x) e^{-2\pi i n x} dx = \int_{\mathbb{R}} f(x) e^{-2\pi i n x} dx = \hat{f}(n)$$

So we get

$$\sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i n x} = g(x) = \sum_{n\in\mathbb{Z}} f(x+n)$$

Taking $x=0$, the Poisson summation formula follows. $\square$

Lemma 9.4 For $x>0$, define the theta series

$$\theta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}.$$

Then $\theta(x)$ satisfies the following modular equation.

$$\theta(x) = x^{-\frac{1}{2}} \theta(x^{-1}).$$

Proof. Consider the function $f(u) = e^{-\pi u^2 x}$. Its Fourier transform is

$$\hat{f}(v) = \int_{\mathbb{R}} e^{-\pi u^2 x} e^{-2\pi i u v} du = \int_{\mathbb{R}} e^{-\pi x (u^2 + \frac{2v}{x} i)} du$$

$$= \int_{\mathbb{R}} e^{-\pi x (u + \frac{v}{x} i)^2 - \frac{\pi v^2}{x}} du$$

$$= e^{-\frac{\pi v^2}{x}} \int_{\mathbb{R}} e^{-\pi x (u + \frac{v}{x} i)^2} du$$

The last integral is the complex integral

$$\int_L e^{-\pi x z^2} dz$$

where $L$ is the infinite segment $(-\infty + \frac{v}{x} i, +\infty + \frac{v}{x} i)$. Note that

$$|e^{-\pi x z^2}| = |e^{-\pi x (\mathrm{Re}\, z + i\, \mathrm{Im}\, z)^2}| = e^{-\pi x ((\mathrm{Re}\, z)^2 - (\mathrm{Im}\, z)^2)}$$

is rapidly decreasing as $|\mathrm{Re}\, z| \to \infty$. So we can move the contour to $(-\infty, +\infty)$, obtaining

$$\int_{\mathbb{R}} e^{-\pi x (u + \frac{v}{x} i)^2} du = \int_{\mathbb{R}} e^{-\pi x u^2} du = \frac{1}{\sqrt{\pi x}} \cdot \sqrt{\pi} = x^{-\frac{1}{2}}.$$

Therefore, we have

$$\hat{f}(v) = x^{-\frac{1}{2}} e^{-\frac{\pi v^2}{x}}.$$

By the Poisson summation formula, we have

$$\theta(x) = \sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) = x^{-\frac{1}{2}} \sum_{n \in \mathbb{Z}} e^{-\frac{\pi n^2}{x}} = x^{-\frac{1}{2}} \theta(x^{-1}).$$

<u>Corollary 7.5</u> Let $\omega(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$. Then we have

$$\omega(x^{-1}) = x^{\frac{1}{2}} \omega(x) + \frac{1}{2}(x^{\frac{1}{2}} - 1).$$

Proof. We have $\theta(x) = 2\omega(x) + 1$. So

$$\omega(x^{-1}) = \frac{1}{2}(\theta(x^{-1}) - 1) = \frac{1}{2}(x^{\frac{1}{2}}\theta(x) - 1)$$

$$= \frac{1}{2}(x^{\frac{1}{2}}\cdot(2\omega(x)+1) - 1) = x^{\frac{1}{2}}\omega(x) + \frac{1}{2}(x^{\frac{1}{2}} - 1). \quad \square$$

Theorem 9.6 For $\mathrm{Re}\, s > 1$, we have

$$\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s) = \int_1^\infty \omega(x)\left(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right)dx + \frac{1}{s(s-1)}.$$

Proof. We begin with

$$\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})n^{-s} = (\pi n^2)^{-\frac{s}{2}}\int_0^\infty e^{-t}t^{\frac{s}{2}}\frac{dt}{t} = \int_0^\infty e^{-t}\left(\frac{t}{\pi n^2}\right)^{\frac{s}{2}}\frac{dt}{t}$$

$$\left(x = \frac{t}{\pi n^2}\right) \quad = \int_0^\infty e^{-\pi n^2 x}x^{\frac{s}{2}}\frac{dx}{x}.$$

Summing over $n$, we get

$$\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s) = \int_0^\infty \omega(x)x^{\frac{s}{2}}\frac{dx}{x} = \left(\int_0^1 + \int_1^\infty\right)\omega(x)x^{\frac{s}{2}}\frac{dx}{x}.$$

By Corollary 9.5, we have

$$\int_0^1 \omega(x)x^{\frac{s}{2}}\frac{dx}{x} = \int_1^\infty \omega(x^{-1})\cdot x^{-\frac{s}{2}}\frac{dx}{x}$$

$$= \int_1^\infty \left(x^{\frac{1}{2}}\omega(x) + \frac{1}{2}(x^{\frac{1}{2}}-1)\right)x^{-\frac{s}{2}}\frac{dx}{x}$$

$$= \int_1^\infty \omega(x)x^{\frac{1-s}{2}}\frac{dx}{x} - \frac{1}{2}\cdot\frac{1}{\frac{1}{2}-\frac{s}{2}} + \frac{1}{2}\cdot\frac{1}{-\frac{s}{2}}$$

$$= \int_1^\infty \omega(x)x^{\frac{1-s}{2}}\frac{dx}{x} - \frac{1}{s(1-s)}$$

Therefore, for $\mathrm{Re}\, s > 1$

$$\pi^{-\frac{s}{2}} \Gamma(\tfrac{s}{2}) \zeta(s) = \int_1^\infty \omega(x) \left( x^{\frac{s}{2}} + x^{\frac{1-s}{2}} \right) \frac{dx}{x} - \frac{1}{s(1-s)}$$ $\square$

Since $\omega(x)$ decreasing rapidly as $x \to +\infty$, the integral

$$\int_1^\infty \omega(x) \left( x^{\frac{s}{2}} + x^{\frac{1-s}{2}} \right) \frac{dx}{x}$$

is absolutely convegent hence defines an entire function.

Corollary 9.7 For $\mathrm{Re}\, s > 1$, define

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma(\tfrac{s}{2}) \zeta(s).$$

Then $\xi(s)$ is holomorphic on $\mathrm{Re}\, s > 1$ and has an analytic continuation to all of $\mathbb{C}$ as a meromorphic function with simple poles at $s=0$ and $s=1$. Moreover,

$$\xi(s) = \xi(1-s) \quad \text{for all } s \in \mathbb{C}.$$ $\square$

Remark. The above analytic continuation of $\xi(s)$ gives that of $\zeta(s)$. The equation

$$\pi^{-\frac{s}{2}} \Gamma(\tfrac{s}{2}) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma(\tfrac{1-s}{2}) \zeta(1-s)$$

is called the functional equation of $\zeta(s)$.

Corollary 9.8 The only zeros of $\zeta(s)$ outside the strip $0 \le \mathrm{Re}\, s \le 1$ are at the negative even integers.

Proof. We know that $\zeta(s)$ has no zero in $\mathrm{Re}\, s > 1$. For $\mathrm{Re}\, s < 0$, by the functional equation, we have

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \zeta(1-s)$$

Note that

- $\mathrm{Re}\, s < 0 \Rightarrow \mathrm{Re}(1-s) > 0$ ;
- $\pi^{s-\frac{1}{2}}$ is entire, $\Gamma(\frac{1-s}{2})$ and $\zeta(1-s)$ are zero-free ;
- $\frac{1}{\Gamma(\frac{s}{2})}$ has zeros at $s = -2, -4, -6, \cdots$ .

The desired result follows. $\square$

The zeros outside the critical strip $0 \le \mathrm{Re}\, s \le 1$ are called the trivial zeros of $\zeta(s)$. The zeta function has infinitely many zeros in the critical strip. These zeros are called the non-trivial zeros of $\zeta(s)$

Proposition 9.4 The non-trivial zeros of $\zeta(s)$ are distributed symmetrically w.r.t. the lines $\mathrm{Re}\, s = \frac{1}{2}$ and $\mathrm{Im}\, s = 0$ .

Proof. It suffices to show the non-trivial zeros of $\zeta(s)$ are distributed symmetricly w.r.t. $s = \frac{1}{2}$ and $\mathrm{Im}\, s = 0$.
The former one follows from the functional equation. To show the latter assertion, it suffices to prove

$$\zeta(\bar{s}) = \overline{\zeta(s)} .$$

This equation is clearly true for $\mathrm{Re}\, s > 1$. So it is true for all $s \in \mathbb{C}$ by the uniqueness of analytic continuation. $\square$

Riemann's Hypothesis Every non-trivial zero of $\zeta(s)$ is on the line $\mathrm{Re}\, s = \frac{1}{2}$ .

We can also obtain some interesting fact from the functional equation of $\zeta(s)$. For example, we have

Proposition 9.10 $\zeta(-1) = -\frac{1}{12}$.

Proof. By the functional equation, we have

$$\zeta(-1) = \pi^{-1-\frac{1}{2}} \frac{\Gamma(1)}{\Gamma(-\frac{1}{2})} \zeta(2) = \frac{\sqrt{\pi}}{6} \cdot \frac{1}{\Gamma(-\frac{1}{2})}$$

By the funtional equation of $\Gamma(s)$, we have

$$\Gamma(-\tfrac{1}{2}) = -2\,\Gamma(\tfrac{1}{2})$$

It remains to calculate $\Gamma(\frac{1}{2})$. We have

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt \xlongequal[dt=2udu]{u=\sqrt{t}} \int_0^\infty \frac{1}{u} e^{-u^2} \cdot 2u\,du$$

$$= 2\int_0^\infty e^{-u^2} du = \int_{-\infty}^{+\infty} e^{-u^2} du = \sqrt{\pi}$$

Thus

$$\zeta(-1) = \frac{\sqrt{\pi}}{6} \cdot \frac{1}{\Gamma(-\frac{1}{2})} = \frac{\sqrt{\pi}}{6} \cdot \frac{1}{-2\Gamma(\frac{1}{2})} = -\frac{1}{12}. \qquad \square$$