Chapter 7

The prime number theorem

In this section, we prove the prime number theorem (Theorem 1.4). The key to the proof is to show that $\zeta(s)$ has no zero on $\operatorname{Re} s = 1$.

7.1 An elementary identity

We begin with an elementary identity.

Lemma 7.1. *For any $\theta \in \mathbb{R}$, we have*

$$3 + 4\cos\theta + \cos 2\theta \geq 0.$$

Proof. We have

$$3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2 \geq 0.$$

□

Theorem 7.2. *For $s = \sigma + it$ with $\sigma > 1$, $t \neq 0$, we have*

$$\operatorname{Re}\left(\frac{3\zeta'(\sigma)}{\zeta(\sigma)} + \frac{4\zeta'(\sigma+it)}{\zeta(\sigma+it)} + \frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right) \geq 0.$$

Proof. We have

$$-\operatorname{Re}\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} = \operatorname{Re}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma+it}} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma}}\operatorname{Re}\left(e^{-it\log n}\right) = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma}}\cos(t\log n).$$

So by Lemma 7.1,

$$\begin{aligned}&-\operatorname{Re}\left(\frac{3\zeta'(\sigma)}{\zeta(\sigma)}+\frac{4\zeta'(\sigma+it)}{\zeta(\sigma+it)}+\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right)\\ &=\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^\sigma}\left(3+4\cos\left(t\log n\right)+\cos\left(2t+\log n\right)\right)\geq 0.\end{aligned}$$

□

7.2 Non-vanishing of $\zeta(s)$ on $\operatorname{Re} s = 1$

Theorem 7.3. *We have $\zeta(1+it)\neq 0$ for any $t\in\mathbb{R}$.*

Proof. Suppose on the contrary that $\zeta(1+it)=0$ for some $t\in\mathbb{R}$. Then $t\neq 0$ since $s=1$ is a pole of $\zeta(s)$. We consider the behaviour of

$$\frac{3\zeta'(\sigma)}{\zeta(\sigma)}+\frac{4\zeta'(\sigma+it)}{\zeta(\sigma+it)}+\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}$$

as $\sigma\to 1^+$.

Since $s=1$ is a simple pole of $\zeta(s)$, we have

$$\frac{\zeta'(\sigma)}{\zeta(\sigma)}=-\frac{1}{\sigma-1}+O(1)$$

as $\sigma\to 1^+$. Suppose $s=1+it$ is a zero of $\zeta(s)$ of order k. Then we have

$$\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}=\frac{k}{\sigma-1}+O(1)$$

as $\sigma\to 1^+$. Finally, we do not know whether $s=1+2it$ is a zero of $\zeta(s)$ or not. But anyway we have

$$\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}=\frac{l}{\sigma-1}+O(1)$$

for some non-negative integer l.

Therefore, as $\sigma\to 1^+$, we have

$$\frac{3\zeta'(\sigma)}{\zeta(\sigma)}+\frac{4\zeta'(\sigma+it)}{\zeta(\sigma+it)}+\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}=\frac{l+4k-3}{\sigma-1}+O(1).$$

But $l + 4k - 3 \geq 4 - 3 \geq 1$. So we have

$$-\operatorname{Re}\left(\frac{3\zeta'(\sigma)}{\zeta(\sigma)} + \frac{4\zeta'(\sigma+it)}{\zeta(\sigma+it)} + \frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right) \to +\infty$$

as $\sigma \to 1^+$. This contradicts Theorem 7.2. □

Actually, Theorem 7.3 is sufficient for proving the prime number theorem. However, in order to not make the proof too technical, we choose to give a quantitative version of Theorem 7.3.

7.3 Lower bound for $\zeta(s)$ near $\operatorname{Re} s = 1$

We first prove a variant of Theorem 7.2.

Theorem 7.4. *For $s = \sigma + it$ with $\sigma > 1$, $t \neq 0$, we have*

$$\zeta^3(\sigma)|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)| \geq 1.$$

Proof. The conclusion is equivalent to

$$\operatorname{Re}\left\{3\log\zeta(\sigma) + 4\log\zeta(\sigma+it) + \zeta(\sigma+2it)\right\} \geq 0.$$

By Taylor's expansion, for $\sigma > 1$, we have

$$\begin{aligned}
\log\zeta(\sigma) &= \log\prod_p\left(1 - \frac{1}{p^s}\right)^{-1} = -\sum_p \log\left(1 - \frac{1}{p^s}\right) \\
&= -\sum_p\sum_{k=1}^{\infty}(-1)^{k+1}\left(-\frac{1}{p^s}\right) = \sum_p\sum_{k=1}^{\infty}\frac{1}{p^{ks}} \\
&= \sum_{n=1}^{\infty}\frac{c_n}{n^s}
\end{aligned}$$

where

$$c_n = \begin{cases} 1, & n \text{ is a prime power,} \\ 0, & \text{otherwise.} \end{cases}$$

Since c_n is non-negative, we can use the same argument as that of Theorem 7.2. □

Now we can prove the key theorem of this chapter:

Theorem 7.5. *There exists a constant $c > 0$ such that*

$$|\zeta(s)| \gg \log^{-7}(|t| + 2)$$

whenever

$$1 - \frac{c}{\log^9(|t| + 2)} < \sigma < 1.$$

Proof. Since $\zeta(1) = \infty$, we can assume that $|t| \neq 0$. Moreover, for $0 < |t| \ll 1$, the conclusion is trivial. So we can assume $|t| \geq 100$. Now we fix t. We first consider $s_1 = \sigma_1 + it$ with

$$\sigma_1 = 1 + \frac{c}{\log^9(|t| + 2)},$$

where $c > 0$ is sufficiently small. By Corollary 6.8, we have

$$\zeta(\sigma_1) = \frac{1}{\sigma_1 - 1} + O(\log(|t| + 2)) \ll \log^9(|t| + 2),$$

$$\zeta(\sigma_1 + 2it) = \frac{1}{\sigma_1 + 2it - 1} + O(\log(|t| + 2)) \ll \log(|t| + 2).$$

So by Theorem 7.4, we have

$$|\zeta(\sigma_1 + it)| \geq \zeta(\sigma_1)^{-\frac{3}{4}} |\zeta(\sigma_1 + 2it)|^{-\frac{1}{4}} \gg \log^{-7}(|t| + 2).$$

Now we consider $\zeta(\sigma_2 + it)$ where

$$\sigma_2 = 1 - \frac{c}{\log^9(|t| + 2)}.$$

Since $|\zeta'(\sigma + it)| \ll \log^2(|t| + 2)$ for $\sigma_1 < \sigma < \sigma_2$ (Corollary 6.8), by the mean value theorem, we still have

$$|\zeta(\sigma_2 + it)| \gg \log^{-7}(|t| + 2)$$

provided that c is sufficiently small. □

Corollary 7.6. *The function $\zeta'(s)/\zeta(s)$ is holomorphic in the region*

$$1 - \frac{c}{\log^9(|t| + 2)} < \sigma < 1, \quad |t| \geq 2.$$

Furthermore, in this region, we have

$$\left| \frac{\zeta'(s)}{\zeta(s)} \right| \ll \log^9(|t| + 2).$$

Proof. It follows from Corollary 6.8 and Theorem 7.5. □

7.4 The prime number theorem

Now we can prove the prime number theorem.

Theorem 7.7 (The prime number theorem)**.** *For $x \geq 2$, we have*

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = x + O\left(x \exp\left(-c(\log x)^{1/10}\right)\right)$$

for some $c > 0$.

Proof. Recall the effective Perron formula (Theorem 5.11):

$$\sum_{n \leq x} \Lambda(n) = \frac{1}{2\pi i} \int_{a-iT}^{a+iT} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) \frac{x^s}{s} \, \mathrm{d}s + O\left(\frac{xA(x)\log x}{T}\right) + O\left(\frac{x^a B(a)}{T}\right)$$

where

$$A(x) = \max_{x/2 \leq n \leq 3x/2} \Lambda(n), \quad B(\sigma) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} = -\frac{\zeta'(\sigma)}{\zeta(\sigma)}.$$

Let $2 \leq T \leq x$ be a parameter specified later. We choose

$$a = 1 + \frac{c}{\log^9 T}$$

where $c > 0$ is the constant in Theorem 7.5 and Corollary 7.6. With this choice of a, we have

$$\frac{x^a B(a)}{T} \ll \frac{x \log^9 T}{T} \exp\left(c \frac{\log x}{\log^9 T}\right)$$

and trivially we have

$$\frac{xA(x)\log x}{T} \ll \frac{x \log^2}{T}.$$

Now we use Cauchy;s theorem to evaluate

$$\frac{1}{2\pi i} \int_{a-iT}^{a+iT} \left(-\frac{\zeta'(s)}{\zeta(s)}\right) \frac{x^s}{s} \, \mathrm{d}s.$$

Let

$$b = 1 - \frac{c}{\log^9 T}.$$

We move the contour to $T_1 \cup T_2 \cup T_3$ where

$$T_1 = [a - iT, b - iT], \quad T_2 = [b - iT, b + iT], \quad T_3 = [b + iT, a + iT].$$

By Corollary 7.6, the only pole in this region is at $s = 1$ with the residue

$$\operatorname{Res}_{s=1} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^s}{s} = \lim_{s \to 1} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) x^s = x.$$

So we have

$$\frac{1}{2\pi i} \int_{a-iT}^{a+iT} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^s}{s} \, \mathrm{d}s = x + \frac{1}{2\pi i} \left\{ \int_{T_1} + \int_{T_2} + \int_{T_3} \right\} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^s}{s} \, \mathrm{d}s.$$

Now x is the desired main term, so it remains to estimate the three integrals:

- On $T_1 \cup T_2$, we have

  $$\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} \ll \frac{x \log^9 T}{T} \exp\left(c \frac{\log x}{\log^9 T} \right)$$

  and

  $$|T_1| + |T_3| \ll \frac{1}{\log^9 T}.$$

  So

  $$\left\{ \int_{T_1} + \int_{T_3} \right\} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^s}{s} \, \mathrm{d}s \ll \frac{x}{T} \exp\left(c \frac{\log x}{\log^9 T} \right).$$

- For the integral on T_2, since now

  $$\sigma = b = 1 - \frac{c}{\log^9 T},$$

  we have

  $$|x^s| = x \exp\left(-c \frac{\log x}{\log^9 T} \right).$$

  Therefore,

  $$\begin{aligned} \int_{T_2} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) \frac{x^s}{s} \, \mathrm{d}s &\ll x \log^9 T \exp\left(-c \frac{\log x}{\log^9 T} \right) \int_{1/2}^{T} \frac{\mathrm{d}t}{t} \\ &\ll x \log^{10} T \exp\left(-c \frac{\log x}{\log^9 T} \right). \end{aligned}$$

In summary, we have shown that

$$\psi(x) = x + O\left(\frac{x\log^2 x}{T}\right) + O\left(x\log^{10} T\exp\left(-c\frac{\log x}{\log^9 T}\right)\right)$$
$$+ O\left(\frac{x\log^9 T}{T}\exp\left(c\frac{\log x}{\log^9 T}\right)\right).$$

Now we specify T by $\log^{10} T = \log x$, i.e.

$$T = \exp\left((\log x)^{1/10}\right).$$

Note that this quantity is larger than any power of $\log x$ but is smaller than any power of x. So $\log x$ can be absorbed by $\exp\left(\alpha(\log x)^{1/10}\right)$ for any $\alpha > 0$. Therefore, with this choice of T, the above error terms can be estimated as follows:

- $\dfrac{x\log^2}{T} = x\log^2\exp(-(\log x)^{1/10}) \ll x\exp\left(-\dfrac{1}{2}(\log x)^{1/10}\right)$.
- $x\log^{10} T\exp\left(-c\dfrac{\log x}{\log^9 T}\right) = x\log x\exp\left(-c(\log x)^{1/10}\right) \ll x\exp\left(-\dfrac{c}{2}(\log x)^{1/10}\right)$.
- $\dfrac{x\log^9 T}{T}\exp\left(c\dfrac{\log x}{\log^9 T}\right) \ll x\log x\exp\left((c-1)(\log x)^{1/10}\right)$.

We can assume that $c < 1$ (since Theorem 7.5 and Corollary 7.6 clearly hold for smaller c), so the last quantity is

$$\ll x\exp\left(-c'(\log x)^{1/10}\right)$$

with $c' = (1-c)/2$. □

Remark. The key point in the identity

$$3 + 4\cos\theta + \cos 2\theta \geq 0$$

is $3 < 4$. By replacing this by other trigonometric identities, we may obtain a better error term in the prime number theorem. However, our method can not get a zero-free region larger than

$$1 - \frac{c}{\log^2(|t|+2)} < \sigma < 1$$

because we have used the mean value theorem and the growth rating of $\zeta'(s)$ is $\log^2|t|$. One may check that this means we could not get an error term better that $O(x\exp(-c(\log x)^{1/3}))$ via this method.