Homework 6

(due Tuesday, April 25, 2023)

1. Calculate all Dirichlet characters modulo 12.

2. This exercise provides an alternative proof of the fact $L(1,\chi) \neq 0$ when χ is real. Let χ be a Dirichlet character modulo q. Define

$$f(n) = \sum_{d|n} \chi(d).$$

   1). Show that $f(n)/\sqrt{n} = (g * h)(n)$ where

$$g(n) = \frac{\chi(n)}{\sqrt{n}}, \quad h(n) = \frac{1}{\sqrt{n}}.$$

   2). Using Dirichlet's hyperbola method, show that

$$\sum_{n \leq x} \frac{f(n)}{\sqrt{n}} = 2L(1,\chi)\sqrt{x} + O(1)$$

   where $\chi \neq \chi_0$.

   3). Suppose that χ is a real (i.e. it only takes values ± 1). Show that $f(1) = 1$ and $f(n) \geq 0$. In addition, show that $f(n) \geq 1$ whenever n is a perfect square.

   4). Deduce from 2) and 3) that $L(1,\chi) \neq 0$ if $\chi \neq \chi_0$ is a real character.