Homework 7

(due Friday, May 5, 2023)

1. Let $v_p(x)$ be the p-adic valuation of $x \in \mathbb{Q}_p$.

   1). Show that if $v_p(x) \neq v_p(y)$,
   $$v_p(x+y) = \inf(v_p(x), v_p(y)).$$

   2). Show that every triangle is isosceles in $\mathbb{Q}_p$.

2. 1). Let p be a prime number and $q \in \mathbb{Q}$ with $v_p(q) \geq 1$. Show that
   $$\sum_{i=0}^{\infty} q^i = \frac{1}{1-q}$$
   in $\mathbb{Q}_p$.

   2). Find the inverse of 6 in $\mathbb{Z}_5$.

3. 1). Find a sequence $\{x_n\}$ in $\mathbb{Q}$ such that $x_n \to 1$ in $\mathbb{R}$ but $x_n \to 0$ in $\mathbb{Q}_2$.

   2). Find a sequence $\{x_n\}$ in $\mathbb{Q}$ such that $x_n \to 1$ in $\mathbb{Q}_3$ but $x_n \to 0$ in $\mathbb{Q}_2$.

4. 1). Show that -1 is a square in $\mathbb{Q}_p$ if and only if $p \equiv 1 \bmod 4$.

   2). Show that -2 is a square in $\mathbb{Q}_p$ if and only if $p \equiv 1, 3 \bmod 8$.

   3). Show that $x^2 + y^2 + z^2 = -2$ is solvable in $\mathbb{Q}_p$ for every prime p.

5. Let $p \neq 2$. Show that there are only 3 quadratic extensions (up to an isomorphism) of $\mathbb{Q}_p$.