Homework 4

(due Friday, April 7, 2023)

1. Show that for $T \geq 2$, $c > 1$ and $x \geq 2$ with $x = N + 1/2$ for some integer N, we have
$$M(x) = \sum_{n \leq x} \mu(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^s}{s\zeta(s)} \, \mathrm{d}s + O\left(\frac{x^c \log x}{T}\right).$$

2. This exercise gives a rigorous proof of Perron's formula via the Laplace transform. We summarize the basics of the Laplace transform in the appendix. Let $D(f;s)$ be a Dirichlet series with σ_a the abscissa of absolute convergence of $D(f;s)$. Let $s \in \mathbb{C}$ such that $\operatorname{Re} s > \sigma_a$. Without loss of generality, we assume that $\sigma_a > 0$.

   1). Show that
   $$D(f;s) = s \int_1^\infty \left(\sum_{n \leq x} f(n)\right) x^{-(s+1)} \, \mathrm{d}x.$$
   Hint: Apply the partial summation formula.

   2). Show that the integral in 1) can be writted as
   $$\int_1^\infty \left(\sum_{n \leq x} f(n)\right) x^{-(s+1)} \, \mathrm{d}x = \int_0^\infty \left(\sum_{n \leq e^t} f(n)\right) e^{-st} \, \mathrm{d}t.$$

   3). Prove Perron's formula: Let $D(f;s)$ be a Dirichlet series. Then for any $c > \sigma_a$ and any non-integer $x > 1$, we have
   $$\sum_{n \leq x} f(n) = \frac{1}{2\pi i} \int_{(c)} D(f;s) \frac{x^s}{s} \, \mathrm{d}s.$$

3. The aim of this exercise is to prove a variant of Perron's formula.

1). Prove that for $c > 0$, we have

$$\frac{1}{2\pi i}\int_{(c)} \frac{x^s}{s^2}\,\mathrm{d}s = \begin{cases} \log x, & x \geq 1, \\ 0, & x \leq 1. \end{cases}$$

2). Let $D(f;s)$ be a Dirichlet series with σ_a the abscissa of absolute convergence of $D(f;s)$. Prove that for any $c > \sigma_a$ and $x \geq 1$, we have

$$\sum_{n\leq x} f(n)\log\frac{x}{n} = \frac{1}{2\pi i}\int_{(c)} D(f;s)\frac{x^s}{s^2}\,\mathrm{d}s.$$

Appendix. The Laplace transform

Let $f(x)$ be a real-valued function defined on $[0,+\infty)$ of bounded variation in any finite interval. Let $s = \sigma + it$ be a complex variable. If the integral

$$\int_0^\infty e^{-sx} f(x)\,\mathrm{d}x = F(s)$$

is absolutely convergent, then for any $x > 0$, we have

$$\lim_{T\to+\infty}\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} e^{xs}F(s)\,\mathrm{d}s = \frac{f(x+)+f(x-)}{2},$$

where $f(x+)$ (resp. $f(x-)$) denotes the right-sided (resp. left-sided) limit of f at x.