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CH12 中心势场问题

(对 Shankar 的 CH12 有重新整理，小节编号和教材中编号不一致)

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{\hat{p}_z^2}{2m} + V(r) \text{，其中 } r = \sqrt{x^2+y^2+z^2}$$

此类 $V(x,y,z) = V(r)$ 的问题统称为中心力场问题或中心势场问题

§12.1 三维欧氏空间的旋转矩阵

回忆 homework 2.

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad R_2(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_3(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

为三维欧氏空间中绕 $x. y. z$ 轴旋转 θ 角的矩阵

我们进一步定义了"生成元"

$$\Omega_1 = i \frac{dR_1}{d\theta}\Big|_{\theta=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\Omega_2 = i \frac{dR_2}{d\theta}\Big|_{\theta=0} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$\Omega_3 = i \frac{dR_3}{d\theta}\Big|_{\theta=0} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

其对易关系为 $[\Omega_1, \Omega_2] = i\Omega_3$, $[\Omega_1, \Omega_3] = -i\Omega_2$

$[\Omega_2, \Omega_3] = i\Omega_1$

$\Rightarrow [\Omega_i, \Omega_j] = i\,\varepsilon_{ijk}\,\Omega_k. \quad (i,j,k = 1,2,3)$

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$$\Omega^2 \equiv \Omega_1^2+\Omega_2^2+\Omega_3^2 = 2I$$

并且我们证明了 $R_i(\theta) = e^{i\theta\Omega_i}$ (Ex. 1.47)

对于中心势场问题，可以利用旋转对称性简化计算

§12.2 Hilbert空间中的角动量算符

经典物理中的角动量为矢量：$\vec{L} = \vec{x}\times\vec{p}$ $\Rightarrow L_i = \varepsilon_{ijk}x^j p^k$

$$L_1 = \varepsilon_{1jk}x^j p^k = x^2p^3 - x^3p^2$$

$$L_2 = \varepsilon_{2jk}x^j p^k = -x^1p^3 + x^3p^1$$

$$L_3 = \varepsilon_{3jk}x^j p^k = x^1p^2 - x^2p^1$$

也常写为 $L_x = yp_z - zp_y$

$$L_y = zp_x - xp_z$$

$$L_z = xp_y - yp_x$$

升级成算符，

$$\hat{L}_x = \hat{Y}\hat{P}_z - \hat{Z}\hat{P}_Y$$

$$\hat{L}_y = \hat{Z}\hat{P}_x - \hat{X}\hat{P}_z$$

$$\hat{L}_z = \hat{X}\hat{P}_Y - \hat{Y}\hat{P}_x$$ 没有算符顺序问题

Ex 12.1 证明 (1) $[\hat{L}_i, \hat{L}_j] = i\hbar\varepsilon_{ijk}\hat{L}_k$

(2) $\hat{L}_i$ 为厄米算符

证明 (1) 用分量形式

$$[\hat{L}_i, \hat{L}_j] = [\varepsilon_{imn}\hat{X}^m\hat{P}^n, \varepsilon_{jkl}\hat{X}^k\hat{P}^l]$$

$$= \varepsilon_{imn}\varepsilon_{jkl}[\hat{X}^m\hat{P}^n, \hat{X}^k\hat{P}^l]$$

$$= \varepsilon_{imn}\varepsilon_{jkl}\{\hat{X}^m[\hat{P}^n, \hat{X}^k]\hat{P}^l + \hat{X}^k[\hat{X}^m, \hat{P}^l]\hat{P}^n\}$$

$$= \varepsilon_{imn}\varepsilon_{kjl}\{\hat{X}^m(-i\hbar)\delta^{kn}\hat{P}^l + \hat{X}^k i\hbar\delta^{ml}\hat{P}^n\}$$

$$= i\hbar(\varepsilon_{imn}\varepsilon_{jkl}\delta^{ml})\hat{X}^k\hat{P}^n - i\hbar(\varepsilon_{imn}\varepsilon_{kjl}\delta^{kn})\hat{X}^m\hat{P}^l$$

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利用 $\varepsilon_{imn}\varepsilon_{jkl}\delta^{ml} = -\varepsilon_{min}\varepsilon_{mjk}$

$$= -(\delta_{ij}\delta_{nk} - \delta_{ik}\delta_{jn})$$

$$\varepsilon_{imn}\varepsilon_{jkl}\delta^{kn} = -\varepsilon_{nim}\varepsilon_{kjl}$$

$$= -(\delta_{ij}\delta_{ml} - \delta_{il}\delta_{mj})$$

$$[\hat{L}_i, \hat{L}_j] = (-i\hbar)(\cancel{\delta_{ij}\delta_{nk}} - \delta_{ik}\delta_{jn})\hat{x}^k\hat{p}^n + i\hbar(\cancel{\delta_{ij}\delta_{ml}} - \delta_{il}\delta_{mj})\hat{x}^m\hat{p}^l$$

$$= i\hbar\,\hat{x}^i\hat{p}^j - i\hbar\,\hat{x}^j\hat{p}^i$$

$$= i\hbar(\hat{x}^i\hat{p}^j - \hat{x}^j\hat{p}^i)$$

另一方面 $\varepsilon_{ijk}\hat{L}_k = \varepsilon_{ijk}\varepsilon_{kmn}\hat{x}^m\hat{p}^n$

$$= \varepsilon_{kij}\varepsilon_{kmn}\hat{x}^m\hat{p}^n$$

$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\hat{x}^m\hat{p}^n$$

$$= \hat{x}^i\hat{p}^j - \hat{x}^j\hat{p}^i$$

$$\Rightarrow [\hat{L}_i, \hat{L}_j] = i\hbar\,\varepsilon_{ijk}\hat{L}_k \quad //$$

- $\hat{L}^i$ $(i=1,2,3)$ 的对易关系和 $\hat{\Omega}^i$ $(i=1,2,3)$ 相同

Ex 12.2 (1) 求 $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$

(2) 证明 $[\hat{L}^2, \hat{L}^i] = 0$

- $[\hat{L}^2, \hat{L}^i] = 0$，这点和 $[\hat{\Omega}^2, \hat{\Omega}^i]$ 也相同

前面证明了 $R_i(\theta) = e^{i\theta\Omega_i}$ 为绕 i 轴转 θ 角，

那么 $e^{\frac{i}{\hbar}\theta\hat{L}_i}$ 有什么物理意义

由于 $\hat{L}_i$ 为 $[x \cdot p]$ 量纲，所以分母有 $\hbar$

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Ex 12.3 在坐标表象下求 $e^{\frac{i}{\hbar}\theta \hat{L}_i}|\psi\rangle$

解: 因为 $\hat{L}_i$ 定义在3维空间，所以其坐标表象为

先考虑 $i=3$

$$\langle xyz|\, e^{\frac{i}{\hbar}\varphi \hat{L}_3}|\psi\rangle$$

$$= \int dx'dy'dz' \,\langle xyz|\, e^{\frac{i}{\hbar}\varphi \hat{L}_3}|x',y',z'\rangle \underbrace{\langle x'y'z'|\psi\rangle}_{\equiv \psi(x',y',z')}$$

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$$\hat{L}_3 = \hat{X}_1 \hat{P}_2 - \hat{X}_2 \hat{P}_1$$

$$\langle xyz | \hat{L}_3 | x'y'z' \rangle$$

$$= \langle xyz | \hat{X}_1 \hat{P}_2 | x'y'z' \rangle - \langle xyz | \hat{X}_2 \hat{P}_1 | x'y'z' \rangle$$

$$= \langle x | \hat{X}_1 | x' \rangle \langle y | \hat{P}_2 | y' \rangle \langle z | z' \rangle - \langle x | \hat{P}_1 | x' \rangle \langle y | \hat{X}_2 | y' \rangle \langle z | z' \rangle$$

$$= x' \delta(x-x') (-i\hbar) \delta(y-y') \frac{\partial}{\partial y'} \delta(z-z')$$

$$- (-i\hbar) \delta(x-x') \frac{\partial}{\partial x'} y' \delta(y-y') \delta(z-z')$$

$$= \delta(x-x') \delta(y-y') \delta(z-z') (-i\hbar) \left[x' \frac{\partial}{\partial y'} - y' \frac{\partial}{\partial x'} \right] \quad \text{--- } (*1)$$

xy 方向转换成极坐标： $\rho' = \sqrt{x'^2 + y'^2}$ ， $\theta' = \arctan \frac{y'}{x'}$

$$x' = \rho' \cos\theta' , \quad y' = \rho' \sin\theta'$$

- $$x' \frac{\partial}{\partial y'} - y' \frac{\partial}{\partial x'} = \rho' \cos\theta' \left[\frac{\partial}{\partial \rho'} \frac{\partial \rho'}{\partial y'} + \frac{\partial}{\partial \theta'} \frac{\partial \theta'}{\partial y'} \right] - \rho' \sin\theta' \left[\frac{\partial}{\partial \rho'} \frac{\partial \rho'}{\partial x'} + \frac{\partial}{\partial \theta'} \frac{\partial \theta'}{\partial x'} \right]$$

$$= \rho' \cos\theta \left[\frac{2y'}{2\sqrt{x'^2 + y'^2}} \frac{\partial}{\partial \rho'} + \frac{\frac{1}{x'}}{1 + \left(\frac{y'}{x'}\right)^2} \frac{\partial}{\partial \theta'} \right]$$

$$- \rho' \sin\theta' \left[\frac{2x'}{2\sqrt{x'^2 + y'^2}} \frac{\partial}{\partial \rho'} + \frac{1}{1 + \left(\frac{y'}{x'}\right)^2} \left(-\frac{y'}{x'^2}\right) \frac{\partial}{\partial \theta'} \right]$$

$$= \rho' \cos\theta' \left[\cancel{\sin\theta' \frac{\partial}{\partial \rho'}} + \frac{1}{\rho'} \cos\theta' \frac{\partial}{\partial \theta'} \right]$$

$$- \rho' \sin\theta' \left[\cancel{\cos\theta' \frac{\partial}{\partial \rho'}} - \frac{1}{\rho'} \sin\theta' \frac{\partial}{\partial \theta'} \right]$$

$$= \left[(\cos\theta')^2 + (\sin\theta')^2 \right] \frac{\partial}{\partial \theta'} = \frac{\partial}{\partial \theta'}$$

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- $\delta(x-x')\delta(y-y') = ?$

利用 $\int dx'dy'\,\delta(x-x')\delta(y-y') = 1$

$$\Rightarrow \int \rho' d\rho' d\theta'\,\delta(x-x')\delta(y-y') = 1$$

$$\Rightarrow \delta(x-x')\delta(y-y') = \frac{1}{\rho'}\delta(\rho-\rho')\delta(\theta-\theta')$$

检查量纲!

所以

$$(*1) \Rightarrow \langle xyz|\hat{L}_3|x'y'z'\rangle = \frac{1}{\rho'}\delta(\rho-\rho')\delta(\theta-\theta')\delta(z-z')(-i\hbar)\frac{\partial}{\partial\theta'} \qquad \text{—— } (*2)$$

$$\langle xyz|\hat{L}_3^2|x'y'z'\rangle = \int dx''dy''dz''\,\langle xyz|\hat{L}_3|x''y''z''\rangle * \langle x''y''z''|\hat{L}_3|x'y'z'\rangle$$

$$= \int \underbrace{dx''dy''}_{=\rho''d\rho''d\theta''}dz''\,\frac{1}{\rho''}\delta(\rho-\rho'')\delta(\theta-\theta'')\delta(z-z'')(-i\hbar)\frac{\partial}{\partial\theta''} * \frac{1}{\rho'}\delta(\rho''-\rho')\delta(\theta''-\theta')\delta(z''-z')(-i\hbar)\frac{\partial}{\partial\theta'}$$

$$= (-i\hbar)^2\frac{1}{\rho}\delta(z-z')\delta(\rho-\rho')\int d\theta''\,\delta(\theta-\theta'')\frac{\partial}{\partial\theta''}\delta(\theta''-\theta')\frac{\partial}{\partial\theta'} \qquad \text{—— } (*3)$$

因为 $\int d\theta'\left[\int d\theta''\,\delta(\theta-\theta'')\frac{\partial}{\partial\theta''}\delta(\theta''-\theta')\frac{\partial}{\partial\theta'}\right]\psi(\theta')$

$$= \int d\theta''\,\delta(\theta-\theta'')\frac{\partial}{\partial\theta''}\underbrace{\int d\theta'\,\delta(\theta''-\theta')\frac{\partial}{\partial\theta'}\psi(\theta')} = \frac{\partial^2\psi}{\partial\theta^2}$$

$$\Rightarrow \int d\theta''\frac{\partial}{\partial\theta''}\delta(\theta''-\theta')\frac{\partial}{\partial\theta'} = \delta(\theta'-\theta'')\frac{\partial^2}{\partial\theta'^2}$$

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代入(*3)即得

$$\langle xyz|\hat{L}_3^{\,2}|x',y'z'\rangle=\frac{1}{\rho}\delta(z-z')\delta(\rho-\rho')\delta(\theta-\theta')(-i\hbar)^2\frac{\partial^2}{\partial\theta'^2}$$

用归纳法可证

$$\langle xyz|\hat{L}_3^{\,n}|x'y'z'\rangle=\frac{1}{\rho}\delta(z-z')\delta(\rho-\rho')\delta(\theta-\theta')(-i\hbar)^n\frac{\partial^n}{\partial\theta'^n}$$

——— (*4)

所以 $\langle xyz|e^{\frac{i}{\hbar}\varphi\hat{L}_3}|x'y'z'\rangle$

$$=\sum_{n=0}^{+\infty}\frac{1}{n!}\left(\frac{i}{\hbar}\varphi\right)^n\langle xyz|\hat{L}_3^{\,n}|x'y'z'\rangle$$

$$=\frac{1}{\rho}\delta(z-z')\delta(\rho-\rho')\delta(\theta-\theta')\sum_{n=0}^{+\infty}\frac{1}{n!}\varphi^n\frac{\partial^n}{\partial\theta'^n}$$

$\langle xyz|e^{\frac{i}{\hbar}\varphi\hat{L}_3}|\psi\rangle$

$$=\int dx'dy'dz'\,\langle xyz|e^{\frac{i}{\hbar}\varphi\hat{L}_3}|x'y'z'\rangle\underbrace{\langle x'y'z'|\psi\rangle}_{=\psi(x',y',z')=\psi(\rho',\theta',z')}$$

$$=\int\rho'd\rho'd\theta'dz'\,\frac{1}{\rho}\delta(z-z')\delta(\rho-\rho')\delta(\theta-\theta')\sum_{n=0}^{+\infty}\frac{1}{n!}\varphi^n\frac{\partial\psi^n}{\partial\theta'^n}$$

$$=\sum_{n=0}^{+\infty}\frac{1}{n!}\varphi^n\frac{\partial\psi^n}{\partial\theta'^n}$$

$$=\psi(\rho,\theta+\varphi,z)$$

令 $|\psi'\rangle=e^{\frac{i}{\hbar}\varphi\hat{L}_3}|\psi\rangle\Rightarrow\psi'(\rho,\theta,z)=\psi(\rho,\theta+\varphi,z)$ //

- $e^{\frac{i}{\hbar}\varphi\hat{L}_3}$ 在坐标空间中将 $\psi(\rho,\theta,z)$ 绕z轴旋转 $-\varphi$ 角

同理 $e^{\frac{i}{\hbar}\varphi\hat{L}_1}$ 和 $e^{\frac{i}{\hbar}\varphi\hat{L}_2}$ 为绕x轴与y轴的相应转动

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- $e^{\frac{i}{\hbar}\varphi \hat{L}_i}$ 也有转动的物理含义，不过是在函数空间中的转动

  $\hat{L}_i$ 即为函数空间中绕 i 轴的无穷小转动

  （回忆：三维欧氏空间中的转动）

- 由 $\hat{L}_i$ 的物理含义，若 $\hat{H} = \frac{\hat{p}^2}{2m} + V(r)$，有

  $$[\hat{L}_i, \hat{H}] = 0$$

  和 $[\hat{L}^2, \hat{H}] = [\hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2, \hat{H}] = 0$

- 一般用 $\hat{H}, \hat{L}^2, \hat{L}_3$ 标记共同的本征态.

Ex 12.4 数学上证明 $[\hat{L}_i, \hat{H}] = 0$

§12.3 $\hat{L}^2$和$\hat{L}_z$的共同本征态

$\hat{L}_i$ 和 $\hat{S}_i$ 虽然具体形式不同，但都有相同的对易关系。在这一节中，我们仅从对易关系出发，看看能得到关于本征值和本征态的哪些信息。

$$[\hat{L}_i, \hat{L}_j] = i\hbar \varepsilon_{ijk} \hat{L}_k$$

$$[\hat{L}^2, \hat{L}_i] = 0$$

• 取 $\hat{L}^2$ 和 $\hat{L}_3$ 的共同本征态 $|\alpha, \beta>$，有

$$\hat{L}^2 |\alpha\beta> = \alpha |\alpha\beta>$$

$$\hat{L}_3 |\alpha\beta> = \beta |\alpha\beta>$$

由于 $[\hat{L}_1, \hat{L}_3] = i\hbar \varepsilon_{132} \hat{L}_2 = -i\hbar \hat{L}_2$

$$[\hat{L}_2, \hat{L}_3] = i\hbar \varepsilon_{231} \hat{L}_1 = i\hbar \hat{L}_1$$

所以该本征态不可能是 $\hat{L}_1$ 和 $\hat{L}_2$ 的本征态。

Ex 12.5 def $\hat{L}_\pm = \hat{L}_1 \pm i\hat{L}_2$

证明 $[\hat{L}_3, \hat{L}_\pm] = \pm\hbar \hat{L}_\pm$

证明: $[\hat{L}_3, \hat{L}_\pm] = [\hat{L}_3, \hat{L}_1] \pm i [\hat{L}_3, \hat{L}_2]$

$$= i\hbar \hat{L}_2 \pm i (-1) i\hbar \hat{L}_1$$

$$= i\hbar \hat{L}_2 \pm \hbar \hat{L}_1 = \pm\hbar(\hat{L}_1 \pm i\hat{L}_2) = \pm\hbar \hat{L}_\pm$$

//

回忆一维谐振子 $[\hat{a}, \frac{\hat{H}}{\hbar\omega}] = \hat{a}$，$[\hat{a}^+, \frac{\hat{H}}{\hbar\omega}] = -\hat{a}^+$

然后我们用梯算符得到了清晰的本征值和本征态结构。

Ex 12.6 (1) 计算 $[\hat{L}_+, \hat{L}_-]$

(2) 计算 $[\hat{L}^2, \hat{L}_\pm]$

解: (1) $$[\hat{L}_+, \hat{L}_-] = [\hat{L}_1 + i\hat{L}_2, \hat{L}_1 - i\hat{L}_2]$$
$$= -i[\hat{L}_1, \hat{L}_2] + i[\hat{L}_2, \hat{L}_1]$$
$$= -i[\hat{L}_1, \hat{L}_2]$$
$$= -i * i\hbar \varepsilon_{123} \hat{L}_3$$
$$= \hbar \hat{L}_3$$

(2) $$[\hat{L}^2, \hat{L}_\pm] = [\hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2, \hat{L}_\pm]$$
$$= [\hat{L}_1^2, \hat{L}_1 \pm i\hat{L}_2] + [\hat{L}_2^2, \hat{L}_1 \pm i\hat{L}_2] + [\hat{L}_3^2, \hat{L}_\pm]$$
$$= \pm i[\hat{L}_1^2, \hat{L}_2] + [\hat{L}_2^2, \hat{L}_1] + [\hat{L}_3^2, \hat{L}_\pm]$$

- $$[\hat{L}_1^2, \hat{L}_2] = \hat{L}_1[\hat{L}_1, \hat{L}_2] + [\hat{L}_1, \hat{L}_2]\hat{L}_1$$
$$= \hat{L}_1 i\hbar \hat{L}_3 + i\hbar \hat{L}_3 \hat{L}_1$$
$$= i\hbar(\hat{L}_1\hat{L}_3 + \hat{L}_3\hat{L}_1)$$
- $$[\hat{L}_2^2, \hat{L}_1] = \hat{L}_2[\hat{L}_2, \hat{L}_1] + [\hat{L}_2, \hat{L}_1]\hat{L}_2$$
$$= \hat{L}_2(-i\hbar)\hat{L}_3 + (-i\hbar)\hat{L}_3\hat{L}_2$$
$$= -i\hbar(\hat{L}_2\hat{L}_3 + \hat{L}_3\hat{L}_2)$$
- $$[\hat{L}_3^2, \hat{L}_\pm] = \hat{L}_3[\hat{L}_3, \hat{L}_\pm] + [\hat{L}_3, \hat{L}_\pm]\hat{L}_3$$
$$= \pm\hbar\hat{L}_3\hat{L}_\pm \pm \hbar\hat{L}_\pm\hat{L}_3$$

$$= \mp\hbar(\cancel{\hat{L}_1\hat{L}_3} + \cancel{\hat{L}_3\hat{L}_1}) - i\hbar(\cancel{\hat{L}_2\hat{L}_3} + \cancel{\hat{L}_3\hat{L}_2})$$
$$\pm\hbar\hat{L}_3(\cancel{\hat{L}_1} \pm \cancel{i\hat{L}_2}) \pm \hbar(\cancel{\hat{L}_1} \pm \cancel{i\hat{L}_2})\hat{L}_3$$
$$= 0$$

下面研究 $\hat{L}_\pm|\alpha\beta\rangle$ 是否是 $\hat{L}^2$ 和 $\hat{L}_3$ 的本征态.

Ex 12.7 计算 $\hat{L}^2\hat{L}_\pm|\alpha\beta\rangle$ 和 $\hat{L}_3\hat{L}_\pm|\alpha\beta\rangle$

解:
$$\hat{L}_3\hat{L}_\pm|\alpha\beta\rangle=\{[\hat{L}_3,\hat{L}_\pm]+\hat{L}_\pm\hat{L}_3\}|\alpha\beta\rangle$$
$$=\{\pm\hbar\hat{L}_\pm+\hat{L}_\pm\hat{L}_3\}|\alpha\beta\rangle$$
$$=\pm\hbar\hat{L}_\pm|\alpha\beta\rangle+\beta\hat{L}_\pm|\alpha\beta\rangle$$
$$=(\beta\pm\hbar)\hat{L}_\pm|\alpha\beta\rangle \qquad (*)$$

$$\hat{L}^2\hat{L}_\pm|\alpha\beta\rangle=\{\cancel{[\hat{L}^2,\hat{L}_\pm]}-\hat{L}_\pm\hat{L}^2\}|\alpha\beta\rangle$$
$$=-\alpha\hat{L}_\pm|\alpha,\beta\rangle \qquad //$$

- 可以证明 $|\alpha\beta\rangle$ 没有简并, 所以
$$\hat{L}_\pm|\alpha\beta\rangle=C_\pm(\alpha,\beta)|\alpha,\beta\pm\hbar\rangle$$

可见 $\hat{L}_\pm$ 可以看成升降算符, 将 $\hat{L}_3$ 的本征值改变 $\hbar$

- 和谐振子类似, 升降不能无休止进行

$\forall\ |\alpha\beta\rangle \qquad \langle\alpha\beta|\hat{L}^2-\hat{L}_3^2|\alpha\beta\rangle=(\alpha-\beta^2)|\alpha\beta\rangle$

另一方面 $\langle\alpha\beta|\hat{L}^2-\hat{L}_3^2|\alpha\beta\rangle=\langle\alpha\beta|\hat{L}_x^2+\hat{L}_y^2|\alpha\beta\rangle$
$$\geqslant 0$$

$\Rightarrow \quad \alpha>\beta^2$

$\Rightarrow$ 给定 α, $\hat{L}_+$ 和 $\hat{L}_-$ 都不能无休止地进行

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Ex 12.8 给定 $\alpha>0$，有 β_{max} 和 β_{min}，满足

$$\hat{L}_+|\alpha,\beta_{max}\rangle = 0$$

$$\hat{L}_-|\alpha,\beta_{min}\rangle = 0$$

研究 $|\alpha\beta\rangle$ 的本征值和本征态.

解：利用 $\hat{L}_-\hat{L}_+ = (\hat{L}_1 - i\hat{L}_2)(\hat{L}_1+i\hat{L}_2) = \hat{L}_1^2+\hat{L}_2^2+i[\hat{L}_1,\hat{L}_2]$

$$= \hat{L}^2-\hat{L}_3^2+i\,i\hbar\hat{L}_3$$

$$= \hat{L}^2-\hat{L}_3^2-\hbar\hat{L}_3$$

可得 $0=\hat{L}_-\hat{L}_+|\alpha\beta_{max}\rangle$

$$= (\hat{L}^2-\hat{L}_3^2-\hbar\hat{L}_3)|\alpha\beta_{max}\rangle$$

$$= (\alpha-\beta_{max}^2-\hbar\beta_{max})|\alpha\beta_{max}\rangle$$

$$\Rightarrow \alpha=\beta_{max}(\beta_{max}+\hbar) \qquad (*1)$$

利用 $\hat{L}_+\hat{L}_- = (\hat{L}_1+i\hat{L}_2)(\hat{L}_1-i\hat{L}_2) = \hat{L}_1^2+\hat{L}_2^2+i[\hat{L}_2,\hat{L}_1]$

$$= \hat{L}^2-\hat{L}_3^2+i(-i\hbar)\hat{L}_3$$

$$= \hat{L}^2-\hat{L}_3^2+\hbar\hat{L}_3$$

可得 $0=\hat{L}_+\hat{L}_-|\alpha\beta_{min}\rangle = (\hat{L}^2-\hat{L}_3^2+\hbar\hat{L}_3)|\alpha\beta_{min}\rangle$

$$= (\alpha-\beta_{min}^2+\hbar\beta_{min})|\alpha\beta_{min}\rangle$$

$$\Rightarrow \alpha=\beta_{min}(\beta_{min}-\hbar) \qquad (*2)$$

比较 (*1) (*2)，可得

$$\beta_{max}^2+\hbar\beta_{max}=\beta_{min}^2-\hbar\beta_{min}$$

$$\Rightarrow (\beta_{max}-\beta_{min})(\beta_{max}+\beta_{min})+\hbar(\beta_{max}+\beta_{min})=0$$

$$\Rightarrow (\underbrace{\beta_{max} - \beta_{min} + \hbar}_{\geq 0})(\beta_{max} + \beta_{min}) = 0$$

$$\Rightarrow \beta_{max} = -\beta_{min} \quad \text{———————} \quad (*3)$$

若从 $|\alpha\beta_{min}\rangle$ 经过 k 次 $\hat{L}_+$ 升到 $|\alpha\beta_{max}\rangle$

可知 $\beta_{max} = \beta_{min} + k\hbar \quad (k=0,1,2\dots)$

结合(*3) $\Rightarrow \beta_{max} = \frac{k\hbar}{2}$ ——————— (*4)

$$\Rightarrow \alpha = \beta_{max}(\beta_{max} + \hbar) = \frac{k\hbar}{2}\left(\frac{k\hbar}{2} + \hbar\right) = \hbar^2 \frac{k}{2}\left(\frac{k}{2}+1\right) \quad \text{———} \quad (*5)$$

总结，$\hat{L}^2$ 和 $\hat{L}_3$ 的共同本征态 $|\alpha\beta\rangle$：

$$\hat{L}^2|\alpha\beta\rangle = \alpha|\alpha\beta\rangle$$

$$\hat{L}_3|\alpha\beta\rangle = \beta|\alpha\beta\rangle$$

其中 α 只能取 $\hbar^2 \frac{k}{2}\left(\frac{k}{2}+1\right) \quad (k=0,1,2\dots)$

给定 α 后，β 只能取 $\underbrace{-\frac{k\hbar}{2}, \left(-\frac{k}{2}+1\right)\hbar \dots \frac{k\hbar}{2}}_{\text{共}(k+1)\text{个}}$

习惯上用 $|l = \frac{k}{2}, m = \frac{\beta}{\hbar}\rangle$ 取代 $|\alpha\beta\rangle$

$$\hat{L}^2|l,m\rangle = l(l+1)\hbar^2|l,m\rangle$$

$$\hat{L}_z|l,m\rangle = m\hbar|l,m\rangle$$

$l = \frac{k}{2}$ 为整数或半整数，$m = -\frac{k}{2}, -\frac{k}{2}+1 \dots \frac{k}{2}$

即为 $m = -l, -l+1 \dots l-1, l$，共 $2l+1$ 个

由 $\hat{L}_\pm|\alpha\beta\rangle = C_\pm(\alpha,\beta)|\alpha,\beta\pm\hbar\rangle$ 可得

$$\hat{L}_\pm|l,m\rangle = C_\pm(l,m)|l,m\pm1\rangle$$

所以 $\hat{L}_\pm$ 只能联系相同 l，但不同 m 的态

Ex 12.9 计算归一化因子 $C_\pm(l,m)$

解：假设 $|l,m\rangle$ 为归一化态矢：$\langle l',m'|l,m\rangle=\delta_{ll'}\delta_{mm'}$

因为 $\hat{L}_\pm=\hat{L}_x\pm i\hat{L}_y$ 互为厄米共轭算符

$$\langle l,m|\hat{L}_+\hat{L}_-|l,m\rangle=|C_-(l,m)|^2\langle l,m-1|l,m-1\rangle$$
$$=|C_-(l,m)|^2$$

又因为 $\hat{L}_+\hat{L}_-=\hat{L}^2-\hat{L}_3^2+\hbar\hat{L}_3$ 见 Ex 12.8

$$\Rightarrow \langle l,m|\hat{L}_+\hat{L}_-|l,m\rangle=\langle l,m|(\hat{L}^2-\hat{L}_3^2+\hbar\hat{L}_3)|l,m\rangle$$
$$=l(l+1)\hbar^2-m^2\hbar^2+m\hbar^2$$
$$=[l(l+1)-m(m-1)]\hbar^2$$

$$\Rightarrow C_-(l,m)=\sqrt{l(l+1)-m(m-1)}\,\hbar$$

自洽性检查

$$\hat{L}_-|l,-l\rangle\propto\sqrt{l(l+1)-(-l)(-l-1)}\,\hbar=0$$

所以 $|l,-l\rangle$ 是“最低”的本征态.

同样地 $\langle l,m|\hat{L}_-\hat{L}_+|l,m\rangle=|C_+(l,m)|^2$

且有 $\langle l,m|\hat{L}_-\hat{L}_+|l,m\rangle=\langle l\,m|\hat{L}^2-\hat{L}_3^2-\hbar\hat{L}_3|l,m\rangle$

$$=l(l+1)\hbar^2-m^2\hbar^2-m\hbar^2$$
$$=[l(l+1)-m(m+1)]\hbar^2$$

$$\Rightarrow C_+(l,m)=\sqrt{l(l+1)-m(m+1)}\,\hbar$$

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自洽性检查 $\hat{L}_+|l,l\rangle \propto \sqrt{l(l+1)-l(l+1)}=0$

所以 $|l,l\rangle$ 是"最高"的本征态 //

总结：在这一节，我们仅利用 $[\hat{L}_i,\hat{L}_j]=i\hbar\varepsilon_{ijk}\hat{L}_k$，即得到了关于本征值和本征态大量的信息

$\hat{L}^2,\hat{L}_3$ 的共同本征态可写成 $|l,m\rangle$ 的形式

$\hat{L}^2|l,m\rangle=l(l+1)\hbar^2|l,m\rangle$

$\hat{L}_3|l,m\rangle=m\hbar|l,m\rangle$

其中 $l\geq0$ 可取整数或半整数

$m=-l,-l+1\ldots l-1,l$ 共 $2l+1$ 个取值

$\hat{L}_\pm=\hat{L}_x\pm i\hat{L}_y$ 可作为升降算符连接相同 l 但不同 m 的态

$\hat{L}_+|l,m\rangle=\sqrt{l(l+1)-m(m+1)}\,\hbar|l,m+1\rangle$

$\hat{L}_-|l,m\rangle=\sqrt{l(l+1)-m(m-1)}\,\hbar|l,m-1\rangle$

$\hat{L}_+|l,l\rangle=0$

$\hat{L}_-|l,-l\rangle=0$

Ex 12.10 三维欧氏空间中的转动矩阵的"生成元" Ω_i 也满足

$$[\Omega_i, \Omega_j] = i\varepsilon_{ijk}\Omega_k$$

所以 $\Omega^2 = \Omega_1^2 + \Omega_2^2 + \Omega_3^2$ 和 Ω_3 的本征矢量也可写成 $|l, m\rangle$ 的形式

(1) 求 l 的取值.

(2) 求 $|l, -l\rangle \cdots |l, l\rangle$ 的列向量形式

(3) 定义 $\Omega_\pm = \Omega_1 \pm i\Omega_2$，写出 $\Omega_\pm$ 的矩阵表示，

并用矩阵计算表明

$$\Omega_+ |l, l\rangle = 0$$

$$\Omega_- |l, -l\rangle = 0$$

$$\Omega_+ |l, m\rangle = \sqrt{l(l+1) - m(m+1)}\, |l, m+1\rangle$$

$$\Omega_- |l, m\rangle = \sqrt{l(l+1) - m(m-1)}\, |l, m-1\rangle$$

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§12.4 角动量算符本征态波函数

在本节中，我们进一步利用角动量算符 $\hat{L}_i$ 的具体形式，

$$\hat{L}_i = \varepsilon_{ijk}\hat{x}_j\hat{p}_k$$

计算 $|l,m\rangle$ 在坐标空间中的波函数.

思路：给定 l，先求 $|l,l\rangle$ 在坐标表象下的波函数，再用 L_- 求其他的 $|l,m\rangle$

Ex 12.11 求 $\hat{L}_3, \hat{L}_1, \hat{L}_2, \hat{L}^2, \hat{L}_\pm$ 在坐标表象下的表示（用球坐标）

解：(1) $\langle xyz|\hat{L}_3|x'y'z'\rangle$ 在 §12.2 节算过z坐标下的表象，现在我们改成球坐标

$$\langle xyz|\hat{L}_3|x'y'z'\rangle = \langle xyz|\hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1|x'y'z'\rangle$$

$$= \langle x|\hat{x}_1|x'\rangle\langle y|\hat{p}_2|y'\rangle\langle z|z'\rangle - \langle x|\hat{p}_1|x'\rangle\langle y|\hat{x}_2|y'\rangle\langle z|z'\rangle$$

$$= \delta(x-x')\delta(y-y')\delta(z-z')\left[x'(-i\hbar)\frac{\partial}{\partial y'} - y'(-i\hbar)\frac{\partial}{\partial x'}\right]$$

$$= \delta(x-x')\delta(y-y')\delta(z-z')(-i\hbar)\left[x'\frac{\partial}{\partial y'} - y'\frac{\partial}{\partial x'}\right]$$

球坐标下

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan\dfrac{\sqrt{x^2+y^2}}{z} \\ \varphi = \arctan\dfrac{y}{x} \end{cases}$$

- $$x'\frac{\partial}{\partial y'} - y'\frac{\partial}{\partial x'} = r'\sin\theta'\cos\varphi'\left[\frac{\partial r'}{\partial y'}\frac{\partial}{\partial r'} + \frac{\partial\theta'}{\partial y'}\frac{\partial}{\partial\theta'} + \frac{\partial\varphi'}{\partial y'}\frac{\partial}{\partial\varphi'}\right] - r'\sin\theta'\sin\varphi'\left[\frac{\partial r'}{\partial x'}\frac{\partial}{\partial r'} + \frac{\partial\theta'}{\partial x'}\frac{\partial}{\partial\theta'} + \frac{\partial\varphi'}{\partial x'}\frac{\partial}{\partial\varphi'}\right]$$

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$$= x'\left\{\cancel{\frac{y'}{r'}\frac{\partial}{\partial r'}} + \cancel{\frac{\frac{1}{z'}\frac{y'}{\sqrt{x'^2+y'^2}}}{1+\frac{x'^2+y'^2}{z'^2}}\frac{\partial}{\partial\theta'}} + \frac{\frac{1}{x'}}{1+\frac{y'^2}{x'^2}}\frac{\partial}{\partial\varphi'}\right\}$$

$$-y'\left\{\cancel{\frac{x'}{r'}\frac{\partial}{\partial r'}} + \cancel{\frac{\frac{1}{z'}\frac{x'}{\sqrt{x'^2+y'^2}}}{1+\frac{x'^2+y'^2}{z'^2}}\frac{\partial}{\partial\theta'}} + \frac{\left(-\frac{y'}{x'^2}\right)}{1+\frac{y'^2}{x'^2}}\frac{\partial}{\partial\varphi'}\right\}$$

$$= \frac{x'^2}{x'^2+y'^2}\frac{\partial}{\partial\varphi'} + \frac{y'^2}{x'^2+y'^2}\frac{\partial}{\partial\varphi'}$$

$$= \frac{\partial}{\partial\varphi'}$$

- $$\int dx'dy'dz'\,\delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

$$= \int r'^2\sin\theta'\,dr'd\theta'd\varphi'\,\delta(r-r')\,\delta(\theta-\theta')\,\delta(\varphi-\varphi')$$

$$\Rightarrow \delta(x-x')\,\delta(y-y')\,\delta(z-z') = \frac{1}{r'^2\sin\theta'}\,\delta(r-r')\,\delta(\theta-\theta')\,\delta(\varphi-\varphi')$$

所以

$$\boxed{\langle x\,y\,z|\hat{L}_3|x',y',z'\rangle = \frac{1}{r'^2\sin\theta'}\,\delta(r-r')\,\delta(\theta-\theta')\,\delta(\varphi-\varphi')\,(-i\hbar)\frac{\partial}{\partial\varphi'}}$$

Ex 12.12 利用 Ex 12.11 的结果，计算 $\langle xyz|ll\rangle$

解：$\langle xyz|\hat{L}_3|ll\rangle = l\hbar\langle xyz|ll\rangle = \psi_{ll}(r,\theta,\varphi)$

另一方面

$$\langle xyz|\hat{L}_3|ll\rangle = \int dx'dy'dz'\langle xyz|\hat{L}_3|x'y'z'\rangle\langle x'y'z'|ll\rangle$$

$$= \int r'^2\sin\theta' dr'd\theta'd\varphi'\frac{1}{r^2\sin\theta}\delta(r-r')\delta(\theta-\theta')\delta(\varphi-\varphi')$$

$$* (-i\hbar)\frac{\partial}{\partial\varphi'}\psi_{ll}(r',\theta',\varphi')$$

$$= (-i\hbar)\frac{\partial}{\partial\varphi}\psi_{ll}(r,\theta,\varphi)$$

$$\Rightarrow (-i\hbar)\frac{\partial}{\partial\varphi}\psi_{ll}(r,\theta,\varphi) = l\hbar\psi_{ll}(r,\theta,\varphi)$$

$$\Rightarrow \psi_{ll}(r,\theta,\varphi) \propto e^{il\varphi}$$

令 $\psi_{ll}(r,\theta,\varphi) = f_{ll}(r,\theta)e^{il\varphi}$，下面求 $f_{ll}(r,\theta)$

$$\langle xyz|\hat{L}_+|ll\rangle = 0$$

$$\Rightarrow \int r'^2\sin\theta' dr'd\theta'd\varphi'\langle xyz|\hat{L}_+|x'y'z'\rangle\langle x'y'z'|ll\rangle$$

$$= \int r'^2\sin\theta' dr'd\theta'd\varphi'\frac{1}{r^2\sin\theta}\delta(r-r')\delta(\theta-\theta')\delta(\varphi-\varphi')$$

$$* \hbar e^{i\varphi'}\left[\frac{\partial}{\partial\theta'} + i\cot\theta'\frac{\partial}{\partial\varphi'}\right]f_{ll}(r',\theta')e^{il\varphi'}$$

$$= \hbar e^{i\varphi}\left[\frac{\partial f_{ll}}{\partial\theta}e^{il\varphi} + i\cot\theta f_{ll}(r,\theta)(il)e^{il\varphi}\right]$$

$$= \hbar e^{i\varphi}e^{il\varphi}\left[\frac{\partial f_{ll}}{\partial\theta} - l\cot\theta f_{ll}(r,\theta)\right] = 0$$

$$\Rightarrow \frac{\partial f_{ll}}{\partial\theta} - l\cot\theta f_{ll}(r,\theta) = 0$$

只与 θ 有关，令 $f_{ll}(r,\theta) = R_{ll}(r)g_{ll}(\theta)$ $\Rightarrow \dfrac{dg_{ll}}{g_{ll}} = \dfrac{l\cos\theta}{\sin\theta}d\theta$

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$$\Rightarrow d\log g_{ll} = l\, d\log\sin\theta \quad \Rightarrow g_{ll} = C_{ll}(\sin\theta)^l$$

$$\Rightarrow \psi_{ll}(r,\theta,\varphi) = C_{ll} R_{ll}(r)(\sin\theta)^l e^{il\varphi}$$

接下来求归一化常数 C_{ll}

$$\int r^2\sin\theta\, dr\, d\theta\, d\varphi\, |\psi_{ll}(r,\theta,\varphi)|^2$$

$$= \int r^2 |R_{ll}(r)|^2 dr * |C_{ll}|^2 \int \sin\theta\, d\theta\, d\varphi\, (\sin\theta)^{2l}$$

若要求径向波函数 $R_{ll}(r)$ 满足 $\int_0^{+\infty} r^2 |R_{ll}(r)|^2 dr = 1$

则有 $|C_{ll}|^2 \int \sin\theta\, d\theta\, d\varphi\, (\sin\theta)^{2l} = 1$

$$\int_0^{\pi} \sin\theta\, d\theta\, d\varphi\, (\sin\theta)^{2l} = (2\pi)\int_{-1}^{1} d(\cos\theta)(1-\cos^2\theta)^l$$

$$= 4\pi\int_0^1 (1-\cos^2\theta)^l d(\cos\theta)$$

$$\overset{x=\cos^2\theta}{=} 4\pi\int_0^1 (1-x)^l \frac{dx}{2\sqrt{x}}$$

$$= 2\pi\int_0^1 (1-x)^l x^{-\frac{1}{2}} dx$$

recall $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx$

$$= 2\pi B\left(\tfrac{1}{2}, l+1\right)$$

$$= 2\pi \frac{\Gamma(\frac{1}{2})\Gamma(l+1)}{\Gamma(l+\frac{3}{2})} = 2\pi \frac{l!}{(l+\frac{1}{2})(l-\frac{1}{2})\cdots\frac{1}{2}}$$

$$= 4\pi \frac{2^l l!}{(2l+1)!!} = 4\pi \frac{(2^l l!)^2}{(2l+1)!}$$

$$\Rightarrow C_{ll} = \sqrt{\frac{4\pi}{(2l+1)!}}\, 2^l l!$$

所以归一化波函数为

$$\psi_{ll}(r,\theta,\varphi) = R_{ll}(r)\sqrt{\frac{4\pi}{(2l+1)!}}\, 2^l l!\, (\sin\theta)^l e^{il\varphi}$$

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recall $Y_{ll}(\theta,\varphi)=(-1)^l\sqrt{\frac{4\pi}{(2l+1)!}}\,2^l l!\,(\sin\theta)^l e^{il\varphi}$

由于 $(-1)^l$ 不影响归一化，所以可以令

$\psi_{ll}(r,\theta,\varphi)=R_{ll}(r)\,Y_{ll}(\theta,\varphi)$

Ex 12.13 利用 Ex 12.11 和 Ex 12.12 的结果，计算 $\langle xyz|l,l-1\rangle$

解: $\langle xyz|\hat{L}_-|x'y'z'\rangle=\frac{1}{r^2\sin\theta}\delta(r-r')\delta(\theta-\theta')\delta(\varphi-\varphi')$

$$*(-\hbar)e^{-i\varphi'}\left[\frac{\partial}{\partial\theta'}-i\cot\theta'\frac{\partial}{\partial\varphi'}\right]$$

$\Rightarrow \langle xyz|\hat{L}_-|l,l\rangle=\int r'^2\sin\theta' dr'd\theta'd\varphi'\frac{1}{r^2\sin\theta}\delta(r-r')\delta(\theta-\theta')\delta(\varphi-\varphi')$

$$*(-\hbar)e^{-i\varphi'}\left[\frac{\partial}{\partial\theta'}-i\cot\theta'\frac{\partial}{\partial\varphi'}\right]$$

$$*(-1)^l\sqrt{\frac{4\pi}{(2l+1)!}}\,2^l l!\,(\sin\theta')^l e^{il\varphi'}*R_{ll}(r)$$

$$=(-\hbar)(-1)^l\sqrt{\frac{4\pi}{(2l+1)!}}\,2^l l!\,e^{-i\varphi}\underbrace{\left[\frac{\partial}{\partial\theta}-i\cot\theta\frac{\partial}{\partial\varphi}\right](\sin\theta)^l e^{il\varphi}}$$

$$*R_{ll}(r)$$

$$=l(\sin\theta)^{l-1}\cos\theta e^{il\varphi}$$
$$-i(\sin\theta)^{l-1}\cos\theta\,(il)e^{il\varphi}$$
$$=2l(\sin\theta)^{l-1}\cos\theta\, e^{il\varphi}$$

$$=(-\hbar)(-1)^l\sqrt{\frac{4\pi}{(2l+1)!}}\,2^l l!\,(2l)(\sin\theta)^{l-1}\cos\theta\, e^{i(l-1)\varphi}$$

$$*R_{ll}(r)$$

另一方面 $\langle xyz|\hat{L}_-|ll\rangle=\sqrt{l(l+1)-l(l-1)}\,\hbar\,\underbrace{\langle xyz|l,l-1\rangle}_{=\psi_{l,l-1}(r,\theta,\varphi)}$

$$=\sqrt{2l}\,\hbar\,\psi_{l,l-1}(r,\theta,\varphi)$$

$\Rightarrow \psi_{l,l-1}(r,\theta,\varphi)=(-1)^{l-1}\sqrt{\frac{4\pi}{(2l+1)!}}\,2^l l!\sqrt{2l}\,(\sin\theta)^{l-1}\cos\theta\, e^{i(l-1)\varphi}$

$$*R_{ll}(r)$$

$$=R_{ll}(r)*Y_{l,l-1}(\theta,\varphi)$$

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所以我们得到所有归一化波函数

$$\langle xyz|lm\rangle = R_{ll}(r)Y_{lm}(\theta,\varphi)$$

其中 $R_{ll}(r)$ 满足 $\int_0^{+\infty} r^2|R_{ll}(r)|^2 dr = 1$

Ex 12.14 考虑 $l=2$ 的态，从 $Y_{22}(\theta,\varphi)$ 开始，用 $\hat{L}_-$ 得到所有 $|2,m\rangle$ 在坐标表象下的波函数，验证得到的结果即为球谐函数

讨论：

(1) 在上一节里，我们推导出 $|l,m\rangle$ 中的 l 可以取 $\frac{k}{2}$ $(k=0,1,2\ldots)$ 而 m 可以取 $-l, -l+1 \ldots l-1, l$

在本节推导中，我们得到 $\psi_{ll}(r,\theta,\varphi) \propto e^{il\varphi}$. 从物理图像上考虑，必须有 $\psi_{ll}(r,\theta,\varphi+2\pi) = \psi_{ll}(r,\theta,\varphi)$. 由此可得 l 必为整数，$m=-l\ldots l$ 也必为整数.

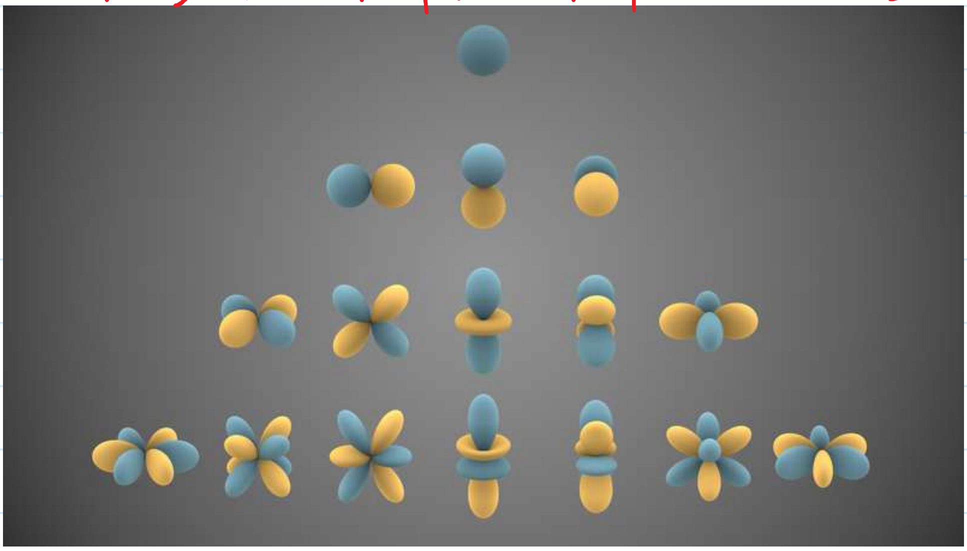
(2) 在 Ex. 12.12 和 Ex 12.13 的推导中，可知 $\psi_{lm}(r,\theta,\varphi)$ 可写成分离变量的形式 $R_{ll}(r)\, g_{lm}(\theta)e^{im\varphi}$ 的形式，每个函数均为一维，因为一维问题没有简并，所以 $|l,m\rangle$ 没有简并

(3) 由 Ex 12.13 的推导，可知径向波函数 $R(r)$ 不依赖于 m 的取值，所以可记 $R_{ll}(r) = R_l(r)$

(4) 球谐函数的几何图像

$m=-3$ $m=-2$ $m=-1$ $m=0$ $m=1$ $m=2$ $m=3$

$l=0$ $l=1$ $l=2$ $l=3$



球谐函数

$m\geq 0$

$$Y_l^m(\theta,\phi)=(-1)^l\left[\frac{(2l+1)!}{4\pi}\right]^{1/2}\frac{1}{2^l l!}\left[\frac{(l+m)!}{(2l)!(l-m)!}\right]^{1/2}e^{im\phi}(\sin\theta)^{-m}\times\frac{d^{l-m}}{d(\cos\theta)^{l-m}}(\sin\theta)^{2l} \tag{12.5.35}$$

对于 $m<0$，有 $Y_{lm}(\theta,\varphi)=(-1)^m\left(Y_{l,-m}(\theta,\varphi)\right)^*$

正交关系为

$$\int d\Omega\, Y_{lm}(\theta,\varphi)\, Y_{l'm'}^*(\theta,\varphi)$$
$$=\int_0^{\pi}d\theta\int_0^{2\pi}d\varphi\, \sin\theta\, Y_{lm}(\theta,\varphi)\, Y_{l'm'}^*(\theta,\varphi)$$
$$=1$$

由 $\psi_{lm}(r,\theta,\varphi)=R_l(r)Y_{lm}(\theta,\varphi)$，可得

$$R_l(r)=\int d\Omega\, \psi_{lm}(r,\theta,\varphi)\, Y_{lm}^*(\theta,\varphi)$$

(5) $|lm\rangle$ 在坐标表示下的波函数也可直接用

$$\hat{L}^2|lm\rangle = l(l+1)\hbar^2|lm\rangle$$

和 $\hat{L}_3|lm\rangle = m\hbar|lm\rangle$

以及 Ex 12.11 的结果得到，见如下练习

Ex 12.15 (1) 用 $\hat{L}_3|lm\rangle = m\hbar|lm\rangle$ 证明

$$\langle xyz|lm\rangle \propto e^{im\varphi}$$

(2) 设 $\langle xyz|lm\rangle = f_{lm}(r,\theta)e^{im\varphi}$，代入

$$\hat{L}^2|lm\rangle = l(l+1)\hbar^2|lm\rangle,$$

求关于 $f_{lm}(r,\theta)$ 的微分方程.

(3) 该微分方程与 r 无关，所以可以令 $f_{lm}(r,\theta) = R_{lm}(r)P_{lm}(\theta)$

求关于 $P_{lm}(\theta)$ 的微分方程

设 $u = \cos\theta$，将 $P_{lm}(\theta)$ 的微分方程中的 θ 换成 u

(4) 令 $P_{lm}(u) = \sum_{n=0}^{+\infty} C_n^{lm} u^n$

代入 (3) 中得到的微分方程，求 C_n^{lm} 的递推关系

(5) 证明 $\lim_{u\to 1} \frac{C_{n+2}^{lm}}{C_n^{lm}} = 1$，所以为了让 $P_{lm}(u)$ 有限，

$P_{lm}(u) = \sum_{n=0}^{+\infty} C_n^{lm} u^n$ 必须在某个 n 处截断.

求出最前面几项的表达式，验证其是否为连带勒让德函数.