Notes on complex analysis

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Textbook: Complex Analysis, Stein & Shakarchi.
References:

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2. Complex Analysis, Ahlfors.

3. Complex variables and applications, Brown& Churchill.

Contents

CHAPTER 1

Preliminaries to complex analysis

1. Notes

1. The geometric meaning of $|f'(z)|^2$. If f is a univalent holomorphic function defined in a region Ω. Then the area of $f(\Omega)$ is

$$\mathrm{Area}(f(\Omega)) = \int_\Omega |f'(z)|^2 \,\mathrm{d}x\mathrm{d}y.$$

2. The mean value theorem in calculus does not hold. The theorem says if $f \in C([a,b])$, then there exits a point $\xi \in (a,b)$ such that

eq:mean

$$f'(\xi) = \frac{f(b)-f(a)}{b-a}. \tag{1}$$

Now we consider the function e^{it} defined on $[0,2\pi]$, which satisfies $e^{i0} = e^{i2\pi} = 1$, but $|(e^{it})'| = |ie^{it}| = 1$. Hence (1) does not hold.

3. The trigonometric functions are unbounded, which is different from the case in $\mathbb{R}$. For instance,

$$\cos z = \frac{e^{iz}+e^{-iz}}{2},$$

if we choose $z = ix$ with $x \in \mathbb{R}$, then $\cos(ix) = \frac{e^x+e^{-x}}{2}$ is unbounded.

4. We consider the exterior differential form for real variables. For $x,y,z \in \mathbb{R}$. The wedge of differentials dx and dy is defined as $dx \wedge dy$, which satisfies

$$dx \wedge dx = 0, \quad dx \wedge dy = -dy \wedge dx.$$

Similarly, we define $dx \wedge dy \wedge dz$.

The exterior differential form ω is the wedge of differentials multiplied by a function. For instance, let F is a function, then F is a exterior differential form of degree zero. Then let A,B,C,P,Q,R,H be functions of x,y,z,

$$\omega = Pdx + Qdy + Rdz$$

is the exterior differential form of degree 1.

$$\omega = Ady \wedge dz + Bdz \wedge dx + Cdx \wedge dy$$

is the exterior differential form of degree 2.

$$\omega = Hdx \wedge dy \wedge dz$$

is the exterior differential form of degree 3.

Then we define the exterior differential operator d on the exterior differential form ω. For $\omega = F$ is a function, we define

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz,$$

which is the total differentiation. For $\omega = Pdx + Qdy + Rdz$, we define

$$d\omega = dP \wedge dx + dQ \wedge dy + dR \wedge dz$$

The we use the definition for dF,

$$\omega = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})dy \wedge dz + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})dz \wedge dx + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dx \wedge dy.$$

Similarly, for $\omega = Ady \wedge dz + Bdz \wedge dx + Cdx \wedge dy$,

$$d\omega = dA \wedge dy \wedge dz + dB \wedge dz \wedge dx + dC \wedge dx \wedge dz = (\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z})dx \wedge dy \wedge dz.$$

If $\omega = Hdx \wedge dy \wedge dz$, we clearly have

$$d\omega = dH \wedge dx \wedge dy \wedge dz = 0.$$

Recall Green's theorem, Stokes theorem and Gauss's theorem.

THEOREM 1 (Green's theorem). *Let Ω be a simply connected domain with piecewise smooth boundary L, and P, $Q \in C^1(\overline{\Omega})$. Then*

$$\int_L P\,\mathrm{d}x + Q\,\mathrm{d}y = \int_\Omega \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,\mathrm{d}x\,\mathrm{d}y.$$

THEOREM 2 (Stokes theorem). *Let Σ be a surface bounded by a piecewise smooth simple closed curve L and P, Q, $R \in C^1(\overline{\Sigma})$. Then*

$$\int_L Pdx + Qdy + Rdz = \int_\Sigma (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})dydz + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})dzdx + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy.$$

THEOREM 3 (Gauss's theorem). *Let Ω be a region bounded by a closed surface Σ, and P, Q, $R \in C^1(\overline{\Omega})$. Then*

$$\int_\Sigma Pdydz + Qdzdx + Rdxdy = \int_\Omega (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z})dxdydz.$$

Hence, we reduce the Green's theorem, Stokes theorem, Gauss's theorem to the uniform formula

$$\int_{\partial\Omega} \omega = \int_\Omega d\omega,$$

which is often called Stokes formula.

5. We now consider the exterior form in $\mathbb{C}$. Consider z and $\bar{z}$ as independent variables. We define the wedge as

$$dz \wedge dz = 0, \quad d\bar{z} \wedge d\bar{z} = 0, \quad dz \wedge d\bar{z} = -d\bar{z} \wedge dz,$$

where

$$dz = dx + idy, \quad d\bar{z} = dx - idy.$$

Then

$$d\bar{z} \wedge dz = 2idx \wedge dy = 2idA,$$

where dA is the area element.

The exterior differential form of degree zero is the function $f(z, \bar{z})$. The exterior differential form of degree 1 is

$$\omega = \omega_1 dz + \omega_2 d\bar{z},$$

where ω_1 and ω_2 are functions of z and $\bar{z}$.The exterior differential form of degree 2 is

$$\omega = \omega_0 dz \wedge d\bar{z},$$

where ω_0 is a function of z and $\bar{z}$.

The exterior differential operator d is defined as

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}.$$

$$d\omega = d\omega_1 \wedge dz + d\omega_2 \wedge d\bar{z} = (\frac{\partial \omega_1}{\partial \bar{z}} - \frac{\partial \omega_2}{\partial z})d\bar{z} \wedge dz.$$

$$d\omega = d\omega_0 d\bar{z} \wedge dz = 0.$$

The we derive the Green's theorem in complex form:

THEOREM 4. *Suppose that $\omega = \omega_1 dz + \omega_2 d\bar{z}$ is an exterior differential form of degree 1, defined on a region Ω, where Ω is bounded by a piecewise smooth curve γ, and ω_1, ω_2 are differentiable functions of z, $\bar{z}$ up to γ. Then*

$$\int_\gamma \omega = \int_\Omega d\omega.$$

2. Exercises

1.

SOLUTION. (a) Midperpendicular of segment $z_1 z_2$.

(b) unit circle.

(c) vertical line with real part 3.

□

2

PROOF. Let $z = x + iy$, $w = u + iv$. Then

$$\langle z, w \rangle = xu + yv.$$

since

$$(z, w) = (x+iy)(u-iv) = xu+yv+i(uy-vx), (w, z) = (u+iv)(x-iy) = ux+vy+i(vx-uy),$$

Thus

$$\langle z, w \rangle = \frac{1}{2}((z, w) + (w, z))\Re(z, w).$$

□

3.

SOLUTION.
$$z = s^{1/n}e^{i\varphi/n} = s^{1/n}e^{i(\varphi/n+2k\pi i)}, \quad \forall k \in \mathbb{N}.$$
□

4

PROOF. Suppose that $i \succ 0$. Then from (iii),
$$-1 \succ 0, -i \succ 0.$$
Then from (ii)
$$0 \succ i,$$
This is contradict to (i). □

5

PROOF. Claim: an open set Ω is pathwise connected iff Ω is connected.

(a) Suppose first that Ω is open and pathwise connected, and that it can be written as $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1 and Ω_2 are disjoint non-empty open sets. Choose two points $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$ and let γ denote a curve in Ω joining w_1 and w_2.

Consider a parametrization $z : [0, 1] \to \Omega$ of this curve with $z(0) = w_1$ and $z(1) = w_2$, and let
$$t^* = \sup_{0\le t\le 1} \{t : z(s) \in \Omega_1, \text{ for all } 0 \le s < t\}.$$

If $z(t^*) \in \Omega_1$, since Ω_1 is open, then there is an open neighborhood of $z(t^*)$ is contained in Ω_1, that is, there exists $\varepsilon > 0$, such that for each $s \in (t^*-\varepsilon, t^*+\varepsilon)$, $z(s)$ is contained in Ω_1, this is contradict to the supremum of t^*. Thus $z(t^*) \in \Omega_2$. But similarly, this is contradict to supermum of t^*.

(b) Suppose that Ω is open and connected. Fix a point $w \in \Omega$ and let $\Omega_1 \subset \Omega$ denote the set of all points that can be joined to w by a curve contained in Ω. Also, let $\Omega_2 \subset \Omega$ denote the set of all points that cannot be joined to w by a curve in Ω.

First, $\Omega_1 \cap \Omega_2 = \emptyset$ is clear.

Now, we prove Ω_1 is open. Choose any point $w_1 \in \Omega_1$, then w_1 is joined to w by a curve γ_1. Since Ω is open, there exists a neighborhood U of w_1 contained in Ω. Clearly, every point in U could be joined to w_1 by a curve γ_2. Then connect the two curves γ_1 and γ_2, thus very point in U can be joined to w by a curve. That is, $U \subset \Omega_1$, hence Ω_1 is open.

Then, we prove Ω_2 is open. Choose any point $w_2 \in \Omega_2$, then there exists a neighborhood of w_2 contained in Ω and very point in this neighborhood is joined to w_2 by a curve γ_3. If there is one point u in this neighborhood does not belong to Ω_2, then there is a curve γ_4 joins w and u, then the curve consists of γ_3 and γ_4 joins w_2 and w, that is $w_2 \in \Omega_1$. This is impossible, since $w_2 \in \Omega_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$.

$\Omega = \Omega_1 \cup \Omega_2$. If not, there exists $v \in \Omega$ and a neighborhood $U(v)$ such that $v \notin \Omega_1 \cup \Omega_2$ and $U(v) \cap \Omega_1 = \emptyset$. Then $v \in \Omega_2$. Contradiction.

Since Ω_1 is empty because of $w \in \Omega_1$, and Ω is connected, thus $\Omega = \Omega_1$. □

7.

PROOF. (a). Let $z = |z|e^{i\theta_1}$, $w = |w|e^{i\theta}$. Then

$$\left|\frac{w-z}{1-\bar{w}z}\right| = \left|\frac{|w|e^{i(\theta_2-\theta_1)} - |z|}{1-|z||w|e^{i(\theta_1-\theta_2)}}\right| \tag{2}$$

Thus, it suffices to assume that $z = r$ is real. We directly compute

$$(r-w)(r-\bar{w}) = r^2 - r(w+\bar{w}) + |w|^2. \tag{3}$$

However,

$$(1-rw)(1-r\bar{w}) = 1 - r(w+\bar{w}) + r^2|w|^2. \tag{4}$$

So

$$(1-rw)(1-r\bar{w}) - (r-w)(r-\bar{w}) = (1-r^2)(1-|w|^2) > 0, \tag{5}$$

since $r < 1$ and $|w| < 1$. In addition,

$$(1-rw)(1-r\bar{w}) - (r-w)(r-\bar{w}) = 0 \Leftrightarrow r = 1 \text{ or } |w| = 1. \tag{6}$$

Hence

$$\left|\frac{w-z}{1-\bar{w}z}\right|^2 \begin{cases} <1, & \text{for } |z|<1 \text{ and } |w|<1 \\ =1, & \text{for } |z|=1 \text{ or } |w|=1 \end{cases} \tag{7}$$

(b). From the above analysis, for $|z| < 1$, $|F(z)| < 1$ and $|z = 1|$, $|F(z)| = 1$. For any $h \in \mathbb{D}$, $h \neq 0$ and $z + h \in \mathbb{D}$, we have

$$\frac{F(z+h)-F(z)}{h} = \frac{|w|^2-1}{(1-\bar{w}z)(1-\bar{w}z-\bar{w}h)} \to \frac{|w|^2-1}{(1-\bar{w}z)^2}, \tag{8}$$

so F is holomorphic. Clearly, $F(0) = w$ and $F(w) = 0$. Moreover, $F \circ F = Id$.

□

8.

PROOF. Let $w = u + iv = f(z) = f(x+iy)$.

$$\begin{aligned}
\frac{\partial h}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) g(u(x,y), v(x,y)) \\
&= \frac{1}{2}\left(\frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x}\right) + \frac{1}{2}\frac{1}{i}\left(\frac{\partial g}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial y}\right) \\
&= \frac{1}{2}\left(\frac{\partial g}{\partial u}\frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial \bar{f}}{\partial x}\right) + \frac{\partial g}{\partial v}\frac{1}{2}\frac{1}{i}\left(\frac{\partial f}{\partial x} - \frac{\partial \bar{f}}{\partial x}\right)\right) \\
&\quad + \frac{1}{2}\frac{1}{i}\left(\frac{\partial g}{\partial u}\frac{1}{2}\left(\frac{\partial f}{\partial y} + \frac{\partial \bar{f}}{\partial y}\right) + \frac{\partial g}{\partial v}\frac{1}{2}\frac{1}{i}\left(\frac{\partial f}{\partial y} - \frac{\partial \bar{f}}{\partial y}\right)\right) \\
&= \frac{1}{2}\frac{\partial g}{\partial u}\left(\frac{\partial f}{\partial z} + \frac{\partial \bar{f}}{\partial z}\right) + \frac{1}{2}\frac{1}{i}\frac{\partial g}{\partial v}\left(\frac{\partial f}{\partial z} - \frac{\partial \bar{f}}{\partial z}\right) \\
&= \frac{\partial g}{\partial w}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{w}}\frac{\partial \bar{f}}{\partial z}.
\end{aligned} \tag{9}$$

□

10.

PROOF. Consider the Laplace operator Δ, we need to define the domain of Δ as $\{f \in C^2\}$. In other words, we need to let the partial derivatives interchange, which is necessary to obtain the equality $\partial_z \partial_{\bar{z}} = \partial_{\bar{z}} \partial_z$. □

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PROOF. Let $f = u + iv$. Then $u = \sqrt{|x||y|}$ and $v = 0$.

$$\partial_x u(0,0) = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = 0, \quad \partial_y u(0,0) = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = 0.$$

Otherwise, $\partial_x v(0,0) = \partial_y v(0,0) = 0$ is trivial. Hence the Cauchy-Riemann equation at the origin. However, for $h = x + iy$,

$$\partial_z f|_{z=0} = \lim_{h \to 0} = \frac{f(z) - f(0)}{h} = \lim_{h = x+iy \to 0} \frac{\sqrt{|x||y|}}{x + iy},$$

which is

$$\begin{cases} \dfrac{1}{1+i}, & \text{when } y = x, x > 0, \\ -\dfrac{1}{1+i}, & \text{when } y = x, x < 0. \end{cases}$$

Thus, f is not holomorphic at 0. □

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PROOF. The partial sum

$$\begin{aligned} S_n &= \frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \cdots + \frac{z^{2^n}}{1-z^{2^{n+1}}} \\ &= \frac{z}{1-z^2} + \left(\frac{1}{1-z^2} - \frac{1}{1-z^4} + \cdots + \left(\frac{1}{1-z^{2^n}} - \frac{1}{1-z^{2^{n+1}}} \right) \right) \\ &= \frac{z}{1-z^2} + \frac{1}{1-z^2} - \frac{1}{1-z^{2^{n+1}}} \\ &\to \frac{1}{1-z} - 1 = \frac{z}{1-z}, \quad \text{as } n \to \infty \text{ and } |z| < 1. \end{aligned}$$

Since

$$\frac{2^k z^{2^k}}{1+z^{2^k}} = \frac{2^k z^{2^k}}{1+z^{2^k}} \frac{1 + z^{2^k} - 2z^{2^k}}{1 - z^{2^k}} = \frac{2^k z^{2^k}}{1 - z^{2^k}} - \frac{2^{k+1} z^{2^{k+1}}}{1 - z^{2^{k+1}}},$$

$$\frac{2^{k+1} z^{2^{k+1}}}{1 - z^{2^{k+1}}} \to 0, \quad \text{as } k \to \infty \text{ and } |z| < 1.$$

Hence, the partial summation

$$\begin{aligned} S_n &= \frac{z}{1+z} + \frac{2z^2}{1+z^2} + \cdots + \frac{2^n z^{2^n}}{1+z^{2^n}} \\ &= \frac{z}{1-z} - \frac{2^{n+1} z^{2^{n+1}}}{1-z^{2^{n+1}}} \\ &\to \frac{z}{1-z}, \quad \text{as } n \to \infty \text{ and } |z| < 1. \end{aligned}$$

$$\frac{1}{1+z} + \frac{2z}{1+z^2} + \cdots + \frac{2^k z^{2^k - 1}}{1+z^{2^k}} + \cdots = \frac{1}{1-z}.$$

□

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PROOF. Assume that $S = \cup_{i=1}^n S_i$. Assign each progression $S_i = \{a_i + kb_i | k \in \mathbb{N}\}$, which generates series

$$\sum_{d=k}^{\infty} z^{a_i + kb_i} = \frac{z^{a_i}}{1 - z^{b_i}} \quad \text{for } |z| < 1.$$

Since S_i, $1 \le i \le n$, partition $\mathbb{N}$,

$$\sum_{i=1}^{n} \sum_{d=k}^{\infty} z^{a_i + kb_i} = \sum_{m=1}^{\infty} z^m = \frac{1}{1-z}, \quad |z| < 1.$$

for this, observe that if $m \in S_i$, then z^m is one of terms being added in $\sum_{d=k}^{\infty} z^{a_i + kb_i}$, and z^m is not in the other series $\sum_{d=k}^{\infty} z^{a_j + kb_j}$ for $j \neq i$. If all the b_i are different, let $b = \max\{b_i\}$, and $\zeta = e^{2\pi i/b}$ be a primitive b-th root of 1. This means $\zeta^b = 1$. If k is an integer, $z^k = 1$ iff k is a multiple of b. If $z^b = 1$, then $z = \zeta^n$ for some integer n. Thus

$$\sum_{k=1}^{n} \frac{z^{a_k}}{1 - z^{b_k}} = \sum_{m} z^m = \frac{z}{1-z},$$

the right side of which tends to $\frac{\zeta}{1-\zeta}$, as $z \to \zeta$. Note that $\zeta \neq 1$ and $b > 1$. On the other hand, if $b_j \neq b$, $\frac{z^{a_k}}{1-z^{b_k}} \to \frac{\zeta^{a_k}}{1-\zeta^{b_k}}$ and $\zeta^{b_j} \neq 1$, since $b_j < b$. BUT if $b_j = b$, then $\frac{z^{a_k}}{1-z^{b_k}} \to \infty$ since $\zeta^{b_j} = \zeta^b = 1$. Thus the left side tends to ∞. This is a contradiction. □

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PROOF.

$$\begin{aligned} \int_\gamma f(z)\,\mathrm{d}z &= \int_a^b f(z(t)) z'(t)\,\mathrm{d}t \\ &= -\int_b^a f(z(t)) z'(t)\,\mathrm{d}t \\ &= \int_{\gamma^-} f(z)\,\mathrm{d}z. \end{aligned} \tag{10}$$

□

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SOLUTION. (a) Let $z = e^{i\theta}$, $\theta \in (-\pi, \pi]$. Then

$$\int_\gamma z^n = \int_{-\pi}^{\pi} ie^{i(n+1)\theta}\,\mathrm{d}\theta = \begin{cases} 2\pi i, & \text{when } n = -1, \\ 0, & \text{otherwise}. \end{cases} \tag{11}$$

(b)

$$\int_\gamma z^n = 0,\ n \in \mathbb{Z}. \tag{12}$$

(c)

$$\int_\gamma \frac{1}{(z-a)(z-b)} = \frac{1}{a-b}\int_\gamma \frac{1}{z-a} - \frac{1}{z-b} = \frac{1}{a-b}(2\pi i - 0) = \frac{2\pi i}{a-b}. \tag{13}$$

□

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PROOF. Suppose that F_1 and F_2 are two primitives of f. Then we have that

$$\frac{d}{dz}(F_1 - F_2) = f'(z) - f'(z) = 0, \tag{14}$$

which along with that $F_1 - F_2$ is holomorphic implies that $F_1 - F_2$ is a constant. □

CHAPTER 2

Cauchy's theorem and its applications

1. Notes

2. Exercises

1.

PROOF. Consider integral of the function e^{iz^2} along the closed contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ defined by

$$\gamma_1 = \{(r,\theta) \in \mathbb{C} : r : 0 \to R,\ \theta = 0\},$$
$$\gamma_2 = \{(r,\theta) \in \mathbb{C} : r = R,\ y : 0 \to \frac{\pi}{4}\},$$

and

$$\gamma_3 = \{(r,\theta) \in \mathbb{C} : r : R \to 0,\ \theta = \frac{\pi}{4}\}.$$

Then we employ Cauchy integral theorem to deduce that

$$0 = \int_0^R e^{ix^2}\,\mathrm{d}x + \int_0^{\frac{\pi}{4}} e^{iR^2 e^{2i\theta}} iRe^{i\theta}\,\mathrm{d}\theta + \int_R^0 e^{ir^2 e^{i\frac{\pi}{2}}} e^{i\frac{\pi}{4}}\,\mathrm{d}r = I + II + III.$$

Since

$$\sin 2\theta \geq \frac{4}{\pi}\theta, \quad \theta \in (0, \pi/4),$$

$$|II| \leq \int_0^{\frac{\pi}{4}} e^{-R^2 \sin 2\theta} R\,\mathrm{d}\theta \leq \int_0^{\frac{\pi}{4}} e^{-R^2 \frac{4}{\pi}\theta} R\,\mathrm{d}\theta = \frac{\pi}{4R}(1 - e^{-R^2}) \to 0,\ R \to \infty.$$

Hence

$$\int_0^\infty e^{ix^2}\,\mathrm{d}x = \int_0^\infty e^{-r^2} e^{i\frac{\pi}{4}}\,\mathrm{d}r = e^{i\frac{\pi}{4}}\frac{\sqrt{\pi}}{2}.$$

which implies the results. □

2

PROOF. Consider the integral of function $\frac{e^{iz}}{z}$ along the toy contour $\gamma = \gamma_1 \cup \gamma_\varepsilon \cup \gamma_2 \cup \gamma_R$ defined by

$$\gamma_1 = \{(r,\theta) \in \mathbb{C} : r : -R \to -\varepsilon,\ \theta = 0\},$$
$$\gamma_\varepsilon = \{(r,\theta) \in \mathbb{C} : r = \varepsilon,\ y : \pi \to 0\},$$
$$\gamma_2 = \{(r,\theta) \in \mathbb{C} : r : \varepsilon \to R,\ \theta = 0\},$$

and

$$\gamma_R = \{(r,\theta) \in \mathbb{C} : r = R,\ \theta : 0 \to \pi\}.$$

Then Cauchy integral theorem implies

$$\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,\mathrm{d}x + \int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\,\mathrm{d}z + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,\mathrm{d}x + \int_{\gamma_R} \frac{e^{iz}}{z}\,\mathrm{d}z = 0.$$

Since

$$\frac{e^{iz}}{z} = \frac{1}{z} + \frac{iz}{z} + E(z),$$

where $E(z)$ is bounded near 0 and $E(z) \to 0$ as $z \to 0$, we have

$$\begin{aligned}\int_{\gamma_\varepsilon} \frac{e^{iz}}{z}\,\mathrm{d}z &= \int_{\pi}^{0} \left(\frac{1}{\varepsilon e^{i\theta}} + i\right) i\varepsilon e^{i\theta}\,\mathrm{d}\theta + \int_{\gamma_\varepsilon} E(z)\,\mathrm{d}z \\ &\to -i\pi, \quad \text{as } \varepsilon \to 0,\end{aligned}$$

since

$$\left|\int_{\gamma_\varepsilon} E(z)\,\mathrm{d}z\right| \le \sup|E(z)|\pi\varepsilon \to 0, \quad \text{as } \varepsilon \to 0.$$

In addition,

$$\left|\int_{\gamma_R} \frac{e^{iz}}{z}\,\mathrm{d}z\right| \le \int_0^{\pi} e^{R\sin\theta}\,\mathrm{d}\theta \le \int_0^{\pi} e^{R\frac{2}{\pi}\theta}\,\mathrm{d}\theta = \frac{\pi}{R}(1 - e^{-R}) \to 0,\ R \to \infty.$$

Since

$$\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,\mathrm{d}x + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,\mathrm{d}x = \int_{\varepsilon}^{R} \frac{e^{ix} - e^{-ix}}{x}\,\mathrm{d}x = 2i\int_{\varepsilon}^{R} \frac{\sin x}{x}\,\mathrm{d}x,$$

Hence

$$2i\int_0^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = i\pi,$$

which is exactly

$$\int_0^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}.$$

□

3.

PROOF. When $b = 0$, these integrals are trivial. Now suppose $b \neq 0$. Consider the integral of function e^{-Az} along the toy contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ defined by

$$\gamma_1 = \{(r,\theta) \in \mathbb{C} : r : 0 \to R,\ \theta = 0\},$$
$$\gamma_2 = \{(r,\theta) \in \mathbb{C} : r = R,\ y : 0 \to \omega\},$$

and

$$\gamma_3 = \{(r,\theta) \in \mathbb{C} : r : R \to 0,\ \theta = \omega\},$$

where

$$A = \sqrt{a^2 + b^2}, \quad \cos\omega = \frac{a}{A}, \quad \sin\omega = \frac{b}{A}.$$

Then the Cauchy integral theorem reveals that

$$\int_0^R e^{-Ax}\,\mathrm{d}x + \int_0^{\omega} e^{-ARe^{i\theta}} iRe^{i\theta}\,\mathrm{d}\theta + \int_R^0 \int_0^{\omega} e^{-Are^{i\omega}} e^{i\omega}\,\mathrm{d}r = 0.$$

Since

$$|II| \leq \int_0^{\omega} e^{-AR\cos\theta} R\,\mathrm{d}\theta \leq \int_0^{\omega} e^{-aR} R\,\mathrm{d}\theta = Re^{-aR}\omega \to 0, \quad R \to \infty,$$

$$\int_0^{\infty} e^{-Ax}\,\mathrm{d}x = e^{i\omega}\int_0^{\infty} e^{-ax-ibx}\,\mathrm{d}x,$$

which implies

$$\int_0^{\infty} e^{-ax}\cos bx\,\mathrm{d}x = \frac{a}{A^2}, \quad \int_0^{\infty} e^{-ax}\sin bx\,\mathrm{d}x = \frac{b}{A^2}.$$

□

4.

PROOF. Note that

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x\xi}\,\mathrm{d}x = e^{-\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi(x-i\xi)^2}\mathrm{d}x. \tag{15}$$

Then we consider the contour $\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, which are defined as

$$\begin{aligned} \gamma_1 &= \{(x,y) \in \mathbb{R}^2 | x : -R \to R,\ y = 0\}, \\ \gamma_2 &= \{(x,y) \in \mathbb{R}^2 | x = R,\ y : R \to R - i\xi\}, \\ \gamma_3 &= \{(x,y) \in \mathbb{R}^2 | x : R - i\xi \to -R - i\xi,\ y = R - i\xi\}, \\ \gamma_4 &= \{(x,y) \in \mathbb{R}^2 | x = -R,\ y : -R - i\xi \to R\}. \end{aligned} \tag{16}$$

We now consider the integral $\int_\Gamma e^{-\pi z^2}\,\mathrm{d}z$. By Cauchy integral theorem,

$$\begin{aligned} 0 &= \int_\Gamma e^{-\pi z^2}\,\mathrm{d}z \\ &= \int_{-R}^{R} e^{-\pi x^2}\,\mathrm{d}x + \int_0^{-\xi} e^{-\pi(R+iy)^2} i\,\mathrm{d}y + \int_R^{-R} e^{-\pi(x-i\xi)^2}\,\mathrm{d}x + \int_{-\xi}^{0} e^{-\pi(-R+iy)^2} i\,\mathrm{d}y \end{aligned} \tag{17}$$

It is evaluated that

$$\begin{aligned} \left|\int_0^{-\xi} e^{-\pi(R+iy)^2} i\,\mathrm{d}y\right| &\leq \left|\int_0^{-\xi} e^{-\pi R^2} e^{-\pi y^2}\,\mathrm{d}y\right| \\ &\leq \int_0^{\infty} e^{-\pi R^2} e^{-piy^2}\,\mathrm{d}y = \frac{1}{2}e^{-\pi R^2} \to 0, \text{ as } R \to \infty. \end{aligned} \tag{18}$$

Similarly,

$$\left|\int_{-\xi}^{0} e^{-\pi(-R+iy)^2} i\,\mathrm{d}y\right| \to 0, \text{ as } R \to \infty. \tag{19}$$

Hence

$$\int_{-\infty}^{\infty} e^{-\pi x^2}\,\mathrm{d}x = \int_{-\infty}^{\infty} e^{-\pi(x-i\xi)^2}\,\mathrm{d}x = 1. \tag{20}$$

□

5.

PROOF. Let $f(z) = u(x,y) + iv(x,y)$. Then $f(z)\,\mathrm{d}z = (u+iv)\,\mathrm{d}x + i(u+iv)\,\mathrm{d}y$. Thus from Green theorem and Cauchy-Riemann equations,

$$\begin{aligned}\int_T f(z)\,\mathrm{d}z &= \int_T u\,\mathrm{d}x - v\,\mathrm{d}y + i\int_T v\,\mathrm{d}x + u\,\mathrm{d}y \\ &= \int_{T_{int}} (-\partial_x v - \partial_y u) + i(\partial_x u - \partial_y v)\,\mathrm{d}x\mathrm{d}y \\ &= 0.\end{aligned} \tag{21}$$

□

6

PROOF. We choose the keyhole contour $\Gamma_{\delta,\varepsilon}$ omitting the point w. The Cauchy integral theorem implies that

$$\int_{\Gamma_{\delta,\varepsilon}} f = 0.$$

Then taking $\delta \to 0$, we have that

$$\int_T f(z)\,\mathrm{d}z = \int_{C_\varepsilon} f(z)\,\mathrm{d}z, \tag{22}$$

where $C_\varepsilon = \{z||z-w| = \varepsilon\}$. From assumption, there exists a constant M such that $|f(z)| \le M$ for $z \in C_\varepsilon$. Thus

$$\left|\int_{C_\varepsilon} f(z)\,\mathrm{d}z\right| \le 2\pi M\varepsilon \tag{23}$$

Then letting $\varepsilon \to 0$ implies

$$\int_T f(z)\,\mathrm{d}z = 0. \tag{24}$$

□

7

PROOF. Since

$$2f'(0) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{f(\zeta) - f(-\zeta)}{\zeta^2}\,\mathrm{d}\zeta \text{ whenever } 0 < r < 1, \tag{25}$$

we have

$$2|f'(0)| \le \frac{1}{2\pi}\int_0^{2\pi} d\frac{1}{r^2}r^2\,\mathrm{d}\theta = d. \tag{26}$$

When $f(z) = a_0 + a_1 z$,

$$d = \sup_{z,w\in\mathbb{D}} |f(z) - f(w)| = |a_1| \sup_{z,w\in\mathbb{D}} |z-w| = 2|a_1|. \tag{27}$$

On the other hand, whenever $0 < r < 1$,

$$2f'(0) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{2a_1}{\zeta}\,\mathrm{d}\zeta = \frac{1}{2\pi i} = 2a_1. \tag{28}$$

□

8

PROOF. For any $x \in \mathbb{R}$, we choose the disk $D_{1/2}(x)$ centered at x with radius $1/2$. Its boundary is the circle $C = C_{1/2}(x)$. Then the Cauchy integral formula reveals that

$$f^{(n)}(x) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - x)^{n+1}}\,\mathrm{d}\zeta.$$

Since

$$|f(\zeta)| \le A(1 + |\zeta|)^\eta,$$

for any ζ in the circle C,

$$|f(\zeta)| \le A(1 + |\zeta - x| + |x|)^\eta \le 2^\eta A(1 + |x|)^\eta.$$

Hence

$$|f^{(n)}(x)| \le \frac{n!}{2\pi}\int_C \frac{2^\eta A(1+|x|)^\eta}{(1/2)^{n+1}}|\,\mathrm{d}\zeta| \le A_n(1 + |x|)^\eta.$$

□

9.

PROOF. We may assume that $z_0 = 0$. Otherwise, we take the function $f(z) = \varphi(z + z_0) - z_0$. Then $f : \Omega - \{z_0\} \to \Omega - \{z_0\}$ is holomorphic and satisfies

$$f(0) = \varphi(z_0) - z_0 = 0, \quad f'(0) = \varphi'(z_0) = 1.$$

If not, we can assume that

$$\varphi(z) = z + a_n z^n + O(z^{n+1})$$

near the origin with $n > 1$ and $a_n \neq 0$. Then by induction, we consider the function

$$\varphi_k(z) = \varphi \circ \cdots \circ \varphi(z) = z + k a_n z^n + O(z^{n+1})$$

Then for $D_\varepsilon(0) \subset \Omega$, and $\varphi_k(\Omega) \subset \Omega$ is holomorphic uniformly for each k, we use the Cauchy inequality to see that

$$|a_n| \le \frac{\varphi_k^{(n)}(0)}{kn!} \le \frac{A}{k\varepsilon^n} \to 0, \quad \text{as } k \to \infty,$$

since A and ε do not depend on k.

□

10.

PROOF. Can every continuous function on the closed unit disk be approximated uniformly by polynomials in the variable of z? **NO**.

The counterexample is $f(z) = \bar{z}$, which is continuous on the closed unit disk. However, $\bar{z}$ can not be approximated by polynomials in the variable of z. The uniform limit of polynomials in the variable of z on the closed disk is a holomorphic function, which is guaranteed by the Weirstrass theorem. □

11.

PROOF. (1). The Cauchy integral formula implies

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{\partial D_R} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\frac{Re^{i\varphi}}{Re^{i\varphi}-z}\,\mathrm{d}\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(Re^{i\varphi})Re^{i\varphi}+z}{Re^{i\varphi}-z}\,\mathrm{d}\varphi - \frac{1}{2\pi}\int_0^{2\pi}\frac{f(Re^{i\varphi})z}{Re^{i\varphi}-z}\,\mathrm{d}\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)\mathrm{d}\varphi \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\frac{1}{2}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z} - \frac{Re^{-i\varphi}+\bar{z}}{Re^{-i\varphi}-\bar{z}}\right)\mathrm{d}\varphi \\
&\quad - \frac{1}{2\pi}\int_0^{2\pi}\frac{f(Re^{i\varphi})z}{Re^{i\varphi}-z}\,\mathrm{d}\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)\mathrm{d}\varphi \\
&\quad - \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\frac{\bar{z}}{Re^{-i\varphi}-\bar{z}}\,\mathrm{d}\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)\mathrm{d}\varphi \\
&\quad - \frac{1}{2\pi}\int_0^{2\pi i} f(Re^{i\varphi})\frac{iRe^{i\varphi}}{\frac{R^2}{\bar{z}}-Re^{i\varphi}}\,\mathrm{d}\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)\mathrm{d}\varphi - \frac{1}{2\pi i}\int_{\partial D_R}\frac{f(\zeta)}{\zeta-\frac{R^2}{\bar{z}}}\,\mathrm{d}\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)\mathrm{d}\varphi.
\end{aligned}
$$

(2).

$$
\mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)
$$

□

12.

PROOF. (a). Let $g(z) = 2\frac{\partial u}{\partial z}$. Since $u \in C^2(\mathbb{D})$, $\mathrm{Re}(g)$ and $\mathrm{Im}(g)$ are continuously differentiable (i.e., $g \in C^1(\mathbb{D})$). In addition,

$$\frac{\partial g}{\partial \bar{z}} = 2\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}u = \frac{1}{2}\Delta u = 0.$$

Hence $g \in H(\mathbb{D})$. Then we might use Goursat' theorem to define the primitive F of f in $\mathbb{D}$ such that $F' = f$. Then

$$\partial_z \mathrm{Re}(F) = \frac{\partial u}{\partial z}$$

implies $Re(F) - u$ is a constant.

□

14.

PROOF. If z_0 is a pole of f with order m, then for z near z_0, we have

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \cdots + \frac{c_{-1}}{z-z_0} + g(z)$$

where $g \in H(\mathbb{D})$. Since $g \in H(\mathbb{D})$, then

$$\begin{aligned} g(z) &= \sum_{n=0}^{\infty} a_n z^n + \frac{c_{-1}}{z_0}\sum_{n=0}^{\infty}\frac{z^n}{z_0^n} + \cdots + (-1)^{m-1}c_{-m}\frac{1}{z_0^m}\sum_{n=0}^{\infty}\frac{z^n}{z_0^n} \\ &= \sum_{n=0}^{\infty}\left(a_n + \frac{c_{-1}}{z_0^{n+1}} + \cdots + (-1)^{m-1}c_{-m}\frac{1}{z_0^{n+m}}\right). \end{aligned}$$

From the convergence of g,

$$a_n + \frac{c_{-1}}{z_0^{n+1}} + \cdots + (-1)^{m-1}c_{-m}\frac{1}{z_0^{n+m}} \to 0, \quad \text{as } n \to \infty.$$

Hence

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}} = z_0.$$

□

15.

PROOF. We employ the maximum principle to see that

$$|f(z)| \le 1, \quad \text{for any } z \in \mathbb{D}.$$

Since f is non-vanishing in $\mathbb{D}$, it is convinced that $\frac{1}{f(z)}$ still satisfies the same conditions as f. Hence, the maximum principle implies that

$$\left|\frac{1}{f(z)}\right| \le 1, \quad \text{for any } z \in \mathbb{D}.$$

Thus $|f(z)| \ge 1$ for any $z \in \mathbb{D}$. Consequently, $|f(z)| = 1$ for any $z \in \mathbb{D}$. The maximum modulus principle guarantees that f is a constant. □

CHAPTER 3

Meromorphic functions and the logarithm

1. Notes

1. Prove that

$$\int_{-\infty}^{\infty} e^{2\pi i x\xi} \frac{\sin \pi a}{\cosh \pi x + \cos \pi a} \, \mathrm{d}x = \frac{\sinh \pi a\xi}{\sinh a\xi},$$

for $0 < a < 1$.

PROOF. Consider the function

$$f(z) = e^{2\pi i z\xi} \frac{\sin \pi a}{\cosh \pi z + \cos \pi a}.$$

Then we choose the contour as $\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, which are defined as

$$\begin{aligned} \gamma_1 &= \{(x,y) \in \mathbb{R}^2 | x : -R \to R, \ y = 0\}, \\ \gamma_2 &= \{(x,y) \in \mathbb{R}^2 | x = R, \ y : 0 \to 2\}, \\ \gamma_3 &= \{(x,y) \in \mathbb{R}^2 | x : R \to -R, \ y = 2\}, \\ \gamma_4 &= \{(x,y) \in \mathbb{R}^2 | x = -R, \ y : 2 \to 0\}. \end{aligned} \tag{29}$$

Since

$$\cosh \pi z + \cos \pi a = \frac{e^{-\pi z}}{2}(e^{2\pi z} + 2e^{\pi z} \cos \pi a + 1) = \frac{e^{-\pi z}}{2}(e^{\pi z} + e^{i\pi a})(e^{\pi z} + e^{-i\pi a}),$$

$f(z)$ has two simple poles at $i(1+a)$ and $i(1-a)$. In addition, the residue of f at $(1-a)i$ is

$$\begin{aligned} res_{z=i(1-a)} f &= 2 \lim_{z \to i(1-a)} e^{2\pi i z\xi} \frac{\sin \pi a(z - i(1-a))}{e^{-\pi z}(e^{\pi z} - e^{i\pi(1+a)})(e^{\pi z} - e^{i\pi(1-a)})} \\ &= 2e^{-2\pi(1-a)\xi} \frac{\sin \pi a}{e^{-i(1-a)\pi} \pi e^{i(1-a)\pi} 2i \sin \pi a} \\ &= \frac{e^{-2\pi(1-a)\xi}}{\pi i}, \end{aligned}$$

and the residue of f at $(1+a)i$ is

$$\begin{aligned} res_{z=i(1+a)} f &= 2 \lim_{z\to i(1+a)} e^{2\pi i z\xi} \frac{\sin \pi a(z-i(1+a))}{e^{-\pi z}\big(e^{\pi z}-e^{i\pi(1+a)}\big)\big(e^{\pi z}-e^{i\pi(1-a)}\big)} \\ &= -2e^{-2\pi(1+a)\xi} \frac{\sin \pi a}{e^{-i(1+a)\pi}\pi e^{i(1+a)\pi} 2i \sin \pi a} \\ &= -\frac{e^{-2\pi(1-a)\xi}}{\pi i}. \end{aligned}$$

The the residue theorem implies that

$$\begin{aligned} &\int_{-R}^{R} e^{2\pi i x\xi} \frac{\sin \pi a}{\cosh \pi x + \cos \pi a}\,\mathrm{d}x + \int_0^2 e^{2\pi i R - 2\pi y} \frac{\sin \pi a}{\cosh \pi(R+iy) + \cos \pi a} i e^{iy}\,\mathrm{d}y \\ &- e^{4\pi\xi} \int_{-R}^{R} e^{2\pi i x\xi} \frac{\sin \pi a}{\cosh \pi x + \cos \pi a}\,\mathrm{d}x - \int_0^2 e^{-2\pi i R - 2\pi y} \frac{\sin \pi a}{\cosh \pi(-R+iy) + \cos \pi a} i e^{iy}\,\mathrm{d}y \\ &= 2\pi i \left(\frac{e^{-2\pi(1-a)\xi}}{\pi i} - \frac{e^{-2\pi(1-a)\xi}}{\pi i} \right) = -4e^{-2\pi\xi} \sinh(2\pi a\xi). \end{aligned}$$

Letting $R \to \infty$,

$$(1-e^{4\pi\xi}) \int_{-\infty}^{\infty} e^{2\pi i x\xi} \frac{\sin \pi a}{\cosh \pi x + \cos \pi a}\,\mathrm{d}x = -4e^{-2\pi\xi} \sinh(2\pi a\xi),$$

which implies

$$\int_{-\infty}^{\infty} e^{2\pi i x\xi} \frac{\sin \pi a}{\cosh \pi x + \cos \pi a}\,\mathrm{d}x = \frac{4e^{-2\pi\xi} \sinh(2\pi a\xi)}{e^{4\pi\xi}-1} = \frac{2\sinh(2\pi a\xi)}{\sinh(2\pi\xi)}.$$

□

2. Exercises

1.

PROOF. From the Euler's formula, we see that

$$sin \pi z = 0 \Leftrightarrow e^{i2\pi z} = 1 \Leftrightarrow z = k \in \mathbb{Z}. \tag{30}$$

By the Taylor's expansion

$$e^{i\pi z} = \sum_{n=0}^{\infty} i^n \pi^n (-1)^k (z-k)^n, \tag{31}$$

we have

$$\sin \pi z = (z-k)\frac{1}{i}\left(i\pi(-1)^k + \sum_{n=1}^{\infty} i^{2n+1}\pi^{2n+1}(-1)^k (z-k)^{2n+1} \right), \tag{32}$$

which implies the zeros are simple. Hence

$$\mathrm{res}_{z=n} \frac{1}{\sin \pi z} = \lim_{z\to n} \frac{z-n}{\sin \pi z} = \frac{(-1)^n}{\pi}. \tag{33}$$

□

2

SOLUTION. Consider the complex function $\frac{1}{1+z^4}$. It has four simple poles $z = e^{\pm i\frac{\pi}{4}}$, $e^{\pm i\frac{3}{4}\pi}$. Then we choose the contour $\Gamma = \gamma_1 \cup \gamma_2$, where

$$\gamma_1 = \{z \in \mathbb{C} | x : -R \to R, \quad y = 0\},$$
$$\gamma_2 = \{z \in \mathbb{C} | |z| = R, \quad argz : 0 \to \pi\}.$$

Then using residue theorem,

$$res_{z=e^{i\frac{\pi}{4}}} f = 2\pi i \lim_{z\to e^{i\frac{\pi}{4}}} \frac{z - e^{i\frac{\pi}{4}}}{1+z^4} = \frac{\pi}{\sqrt{2}(1+i)}, \tag{34}$$

and

$$res_{z=e^{i\frac{3}{4}\pi}} f = 2\pi i \lim_{z\to e^{i\frac{3}{4}\pi}} \frac{z - e^{i\frac{3}{4}\pi}}{1+z^4} = \frac{\pi}{\sqrt{2}(1-i)}. \tag{35}$$

Thus

$$\int_\Gamma \frac{1}{1+z^4}\,\mathrm{d}z = \int_{\gamma_1} + \int_{\gamma_2} = \frac{\pi}{\sqrt{2}}. \tag{36}$$

Otherwise, by Cauchy integral theorem

$$\int_\Gamma \frac{1}{1+z^4}\,\mathrm{d}z = \int_{-R}^{R} \frac{1}{1+x^4}\,\mathrm{d}x + \int_0^\pi \frac{Rie^{i\theta}}{1+R^4e^{i4\theta}}\,\mathrm{d}\theta. \tag{37}$$

Since

$$\left|\int_0^\pi \frac{Rie^{i\theta}}{1+R^4e^{i4\theta}}\,\mathrm{d}\theta\right| \to 0, \quad \text{as } R \to \infty,$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,\mathrm{d}x = \frac{\pi}{\sqrt{2}}. \tag{38}$$

□

3.

PROOF. Consider the function

$$f(z) = \frac{e^{iz}}{z^2+a^2}.$$

Then we choose the contour $\Gamma = [-R, R] \cup C_R$ with positive orientation, where $C_R = \{z \in \mathbb{C} | \Im z \geq 0, \ |z| = R\}$ and $R > 2a$. Then $f(z)$ has a simple pole at ia in the interior of Γ. The residue of f at $z = ia$ is

$$\mathrm{res}_{z=ia} f = \lim_{z\to ia} (z - ia)\frac{e^{iz}}{z^2+a^2} = \frac{e^{-a}}{2ia}.$$

The residue theorem implies that

$$\int_{-R}^{R} \frac{e^{ix}}{x^2+a^2}\,\mathrm{d}x + \int_0^\pi \frac{e^{Re^{i\theta}}}{R^2e^{2i\theta}+a^2} Rie^{i\theta}\,\mathrm{d}\theta = \pi\frac{e^{-a}}{a}.$$

We estimate

$$\left|\int_0^\pi \frac{e^{Re^{i\theta}}}{R^2e^{2i\theta}+a^2} Rie^{i\theta}\,\mathrm{d}\theta\right| \le \int_0^\pi \frac{R}{R^2-a^2}\,\mathrm{d}\theta \le \frac{2\pi}{R} \to 0$$

as $R\to\infty$. Finally, let $R\to\infty$ and take the real part to deduce that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,\mathrm{d}x = \pi\frac{e^{-a}}{a}.$$

□

4.

PROOF. Consider

$$f(z) = \frac{ze^{iz}}{z^2+a^2}.$$

Then we choose the contour $\Gamma = [-R,R]\cup C_R$ with positive orientation, where $C_R = \{z\in \mathbb{C} | \Im z \ge 0,\ |z| = R\}$ and $R > 2a$. Then $f(z)$ has a simple pole at ia in the interior of Γ. The residue of f at $z = ia$ is

$$\mathrm{res}_{z=ia} f = \lim_{z\to ia}(z-ia)\frac{ze^{iz}}{z^2+a^2} = e^{-a}.$$

The residue theorem implies that

$$\int_{-R}^{R} \frac{xe^{ix}}{x^2+a^2}\,\mathrm{d}x + \int_0^\pi \frac{Re^{i\theta}e^{Re^{i\theta}}}{R^2e^{2i\theta}+a^2} Rie^{i\theta}\,\mathrm{d}\theta = \pi i e^{-a}.$$

We estimate

$$\left|\int_0^\pi \frac{Re^{i\theta}e^{Re^{i\theta}}}{R^2e^{2i\theta}+a^2} Rie^{i\theta}\,\mathrm{d}\theta\right| \le \int_0^\pi \frac{R^2e^{-R\sin\theta}}{R^2-a^2}\,\mathrm{d}\theta \le 2\frac{R^2}{R^2-a^2}\int_0^{\pi/2} e^{-2R\theta/\pi}$$

$$= \frac{R^2}{R^2-a^2}\frac{\pi}{R}(1-e^{-R}) \to 0$$

as $R\to\infty$. Finally, let $R\to\infty$ and take the imaginary part to deduce that

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}\,\mathrm{d}x = \pi e^{-a}.$$

□

5.

PROOF. Consider the function

$$f(z) = \frac{e^{2\pi i z\xi}}{(1+z^2)^2},$$

(1). For $\xi \geq 0$, we choose the contour $\Gamma = [-R, R] \cup C_R$ with positive orientation, where $C_R = \{z \in \mathbb{C} | \Im z \geq 0,\ |z| = R\}$ and $R > 2$. Then $f(z)$ has a pole of order 2 at i in the interior of Γ. The residue of f at $z = i$ is

$$\mathrm{res}_{z=i} f = \lim_{z \to i} \frac{d}{dz}(z-i)^2 \frac{e^{2\pi i z\xi}}{(z+i)^2(z-i)^2} = \pi\xi \frac{e^{-2\pi\xi}}{2i} + \frac{e^{-2\pi\xi}}{4i}.$$

The residue theorem implies that

$$\int_{-R}^{R} \frac{e^{2\pi i x\xi}}{(1+x^2)^2}\,\mathrm{d}x + \int_0^{\pi} \frac{e^{2\pi i\xi Re^{i\theta}}}{(R^2e^{2i\theta}+1)^2} Rie^{i\theta}\,\mathrm{d}\theta = \frac{\pi}{2}(1+2\pi\xi)e^{-2\pi\xi}.$$

We estimate

$$\left|\int_0^{\pi} \frac{e^{2\pi i\xi Re^{i\theta}}}{(R^2e^{2i\theta}+1)^2} Rie^{i\theta}\,\mathrm{d}\theta\right| \leq \int_0^{\pi} \frac{R}{(R^2-1)^2}\,\mathrm{d}\theta \leq \frac{2\pi}{R^3} \to 0$$

as $R \to \infty$. Finally, let $R \to \infty$ and take the real part to deduce that

$$\int_{-\infty}^{\infty} \frac{e^{2\pi i x\xi}}{(1+x^2)^2}\,\mathrm{d}x = \frac{\pi}{2}(1+2\pi\xi)e^{-2\pi\xi}.$$

(2). For $\xi < 0$, we choose the contour $\Gamma = [-R, R] \cup C_R$ with positive orientation, where $C_R = \{z \in \mathbb{C} | \Im z \leq 0,\ |z| = R\}$ and $R > 2$. Then $f(z)$ has a pole of order 2 at $-i$ in the interior of Γ. The residue of f at $z = -i$ is

$$\mathrm{res}_{z=-i} f = \lim_{z \to -i} \frac{d}{dz}(z+i)^2 \frac{e^{2\pi i z\xi}}{(z+i)^2(z-i)^2} = \pi\xi \frac{e^{2\pi\xi}}{2i} - \frac{e^{2\pi\xi}}{4i}.$$

The residue theorem implies that

$$-\int_{-R}^{R} \frac{e^{2\pi i x\xi}}{(1+x^2)^2}\,\mathrm{d}x + \int_{-\pi}^{0} \frac{e^{2\pi i\xi Re^{i\theta}}}{(R^2e^{2i\theta}+1)^2} Rie^{i\theta}\,\mathrm{d}\theta = \frac{\pi}{2}(-1+2\pi\xi)e^{2\pi\xi}.$$

We estimate

$$\left|\int_{-\pi}^{0} \frac{e^{2\pi i\xi Re^{i\theta}}}{(R^2e^{2i\theta}+1)^2} Rie^{i\theta}\,\mathrm{d}\theta\right| \leq \int_{-\pi}^{0} \frac{R}{(R^2-1)^2}\,\mathrm{d}\theta \leq \frac{2\pi}{R^3} \to 0$$

as $R \to \infty$. Finally, let $R \to \infty$ and take the real part to deduce that

$$\int_{-\infty}^{\infty} \frac{e^{2\pi i x\xi}}{(1+x^2)^2}\,\mathrm{d}x = \frac{\pi}{2}(1-2\pi\xi)e^{-2\pi\xi}.$$

$\square$

6.

PROOF. Consider the function

$$f(z) = \frac{1}{(1+z^2)^{n+1}}$$

with poles at $z = \pm i$ of order $n+1$. Then the residue of f at $z = i$ is

$$res_{z=i} f = \frac{1}{n!} \lim_{z \to i} \frac{d^n}{dz^n} (z-i)^{n+1} \frac{1}{(1+z^2)^{n+1}} = \frac{(n+1)\cdot \cdots \cdot 2n}{n!} \frac{1}{2^{2n+1} i}.$$

Then we choose the contour $\Gamma = \gamma_1 \cup \gamma_2$, where

$$\gamma_1 = \{z \in \mathbb{C} | x : -R \to R, \quad y = 0\},$$
$$\gamma_2 = \{z \in \mathbb{C} | |z| = R, \quad arg z : 0 \to \pi\}.$$

By the residue formula,

$$\int_{-R}^{R} \frac{1}{(1+x^2)^{n+1}} \, \mathrm{d}x + \int_0^{\pi} \frac{1}{(1+Re^{i\theta})^{n+1}} Rie^{i\theta} \, \mathrm{d}\theta = 2\pi i res_{z=i} f = \frac{(2n-1)!!}{(2n)!!} \pi.$$

Since

$$\left| \int_0^{\pi} \frac{1}{(1+R^2 e^{2i\theta})^{n+1}} Rie^{i\theta} \, \mathrm{d}\theta \right| \geq \int_0^{\pi} \frac{R}{(R^2-1)^{n+1}} \, \mathrm{d}\theta \to 0, \quad \text{as } R \to \infty,$$

letting $R \to \infty$ implies the result. □

7.

PROOF. Consider the function

$$f(z) = \frac{1}{iz} \frac{1}{\left(a + \frac{1}{2}(z + \frac{1}{z})\right)^2} = \frac{4z}{i(z^2 + 2az + 1)^2}$$

which has a pole of order 2 at

$$z_0 = -a + \sqrt{a^2 - 1} \in \mathbb{D}.$$

Since

$$\begin{aligned} res_{z=z_0} f &= \lim_{z \to z_0} \frac{d}{dz} \frac{4z}{i(z + a + \sqrt{a^2-1})^2} \\ &= \frac{1}{i(a^2-1)} - \frac{-a + \sqrt{a^2-1}}{i(a^2-1)^{3/2}} \\ &= \frac{a}{i(a^2-1)^{3/2}}, \end{aligned}$$

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{(a^2 + \cos^2\theta)^2} = \int_C f(z) \, \mathrm{d}z = \frac{2\pi a}{(a^2-1)^{3/2}}.$$

□

REMARK. We consider the definite integrals of the type

$$\int_0^{2\pi} f(\sin\theta, \cos\theta) \, \mathrm{d}\theta.$$

we use

$$z = e^{i\theta}, \quad 0 \leq \theta \leq 2\pi$$

to denote the unit circle C. Then we have

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right), \quad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \mathrm{d}\theta = \frac{\mathrm{dz}}{iz}.$$

Hence

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)\,\mathrm{d}\theta = \int_C f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right)\frac{\mathrm{dz}}{iz}.$$

□

8.

PROOF. Consider the function

$$f(z) = \frac{1}{iz}\frac{1}{a + b\frac{z+z^{-1}}{2}} = \frac{2}{i(bz^2 + 2az + b)},$$

which has a simple pole at $z_0 = \frac{-a+\sqrt{a^2-b^2}}{b}$ in the unit disk $\mathbb{D}$. Then

$$res_{z=z_0} f = \frac{b}{2i\sqrt{a^2 - b^2}}.$$

Then from residue theorem

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\cos\theta} = \int_C f(z)\,\mathrm{d}z = \frac{\pi b}{\sqrt{a^2 - b^2}}.$$

□

10.

PROOF. Let $f(z) = \frac{\log|z|}{z^2+a^2}$. We have that

$$\left|\int_{C_\varepsilon} f(z)\right| \leq \pi\frac{\varepsilon\log(-\varepsilon)}{a^2 - \varepsilon^2} \to 0, \quad \varepsilon \to 0, \tag{39}$$

and

$$\left|\int_{C_R} f(z)\right| \leq \pi\frac{R\log R}{R^2 - a^2} \to 0, \quad R \to \infty. \tag{40}$$

Finally, the residue formula implies the result. □

11.

PROOF. Consider the function $\log z$ in the disk $D_{a'}(1)$ centered at 1 with radius a' such that $|a| < a' < 1$. Then $\log z \in D_{a'}(1)$. The real part satisfies

$$0 = \log 1 = \frac{1}{2\pi}\int_0^{2\pi} \log|1 + ae^{i\theta}| \quad \text{for } a > 0.$$

$a = 0$ is trivial. For $a < 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} \log|e^{-i\pi} + |a|e^{i\theta}| = \Re\log(e^{i\pi}) = 0.$$

□

12.

PROOF. Choose the circle $C := |z| = N + \frac{1}{2}$ such that $N > |u|$ and N is an integer. Then in the interior of this circle, $f(z)$ has a pole of second order at $z = -u$ and a simple pole at the integer n with $|n| \leq N$. Then from the residue formula,

$$0 = \frac{1}{2\pi i} \int_C f(z)\,\mathrm{d}z = -\frac{\pi^2}{\sin(\pi u)^2} + \sum_{n=-N}^{N} \frac{1}{(u+n)^2},$$

since $\cot \pi z = 0$ on the circle. Then letting $N \to \infty$ yields the result. □

13.

PROOF. Let $g(z) = (z - z_0)f(z)$. Then from $|g(z)| \leq A|z - z_0|^{\varepsilon}$, we know that z_0 is a simple zero of $g(z)$ and g is holomorphic. Then the Taylor expansion of $g(z)$ at $z = z_0$

$$g(z) = g'(z_0)(z - z_0) + \frac{g''(z_0)}{2}(z - z_0)^2 + \ldots = (z - z_0)f(z)$$

reveals that

$$f(z) = \sum_{n=1}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z - z_0)^{n-1}$$

is holomorphic and $f(z_0) = g'(z_0)$. □

16.

PROOF. (a). We choose $\varepsilon < \min_{|z|=1} \left|\frac{f(z)}{g(z)}\right|$, so that $|f(z)| > \varepsilon|g(z)|$ on $|z| = 1$. This allows us to use Rouće theorem to obtain the uniqueness of zero of $f_{\varepsilon}(z)$ in $|z| \leq 1$.

(b). Since 0 is a simple zero of $f(z)$, there exists a holomorphic function $h(z)$ nowhere vanishes in $|z| \leq 1$ so that $f(z) = zh(z)$. Since

$$f_{\varepsilon}(z_{\varepsilon}) = z_{\varepsilon}h(z_{\varepsilon}) + \varepsilon g(z_{\varepsilon}) = 0,$$

$$|z_{\varepsilon}| = \left|\varepsilon \frac{g(z_{\varepsilon})}{h(z_{\varepsilon})}\right| \leq \varepsilon \max_{|z=1|} \left|\frac{g(z)}{h(z_)}\right|$$

implies that $\varepsilon \mapsto z_{\varepsilon}$ is continuous. □

17.

PROOF. (a) It suffices to show that if $|w| < 1$, $g(z) = f(z) - w$ has a zero in $\mathbb{D}$. It is easy to see that

$$|g(z) - f(z)| = |w| < 1 = |f(z)| \quad \text{on } |z| = 1.$$

So by Rouché's theorem, f and g have the same number of zeros in $\mathbb{D}$. So it suffices to show that f has a zero in $\mathbb{D}$.

From the hypothesis of f, we employ maximum modulus principle to see that there exists a $z_0 \in \mathbb{D}$ such that $f(z_0) \in \mathbb{D}$. Let $h(z) = f(z) - f(z_0)$ so that

$$|h(z) - f(z)| = |f(z_0)| < 1 = |f(z)| \quad \text{on } |z| = 1.$$

We again use Rouché's theorem to see that f and h have the same number of zeros in $\mathbb{D}$. Since h has at least one zero in $\mathbb{D}$, f has at least one zero in $\mathbb{D}$.

(b) The proof is similar as part (a), after a slight modification.

□

22.

PROOF. Assume that f is holomorphic in $\mathbb{D}$ and $f \in C(\bar{\mathbb{D}})$. If $f(z) = \frac{1}{z}$ on $\partial\mathbb{D}$, then the Cauchy integral theorem yields

$$0 = \int_{\partial\mathbb{D}} f(z)\,\mathrm{d}z = \int_{\partial\mathbb{D}} \frac{1}{z}\,\mathrm{d}z = 2\pi i,$$

which is a contradiction. □

CHAPTER 4

The Fourier transform

1.

PROOF. (a)

$$A(\xi) - B(\xi) = e^{2\pi i\xi t} \int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}\,\mathrm{d}x = e^{2\pi i\xi t}\hat{f}(\xi) = 0.$$

(b) The Schwarz reflection principle guarantees that F is holomorphic, and hence entire. Then F is bounded, since f is moderate decreasing. By Liouville's theorem, F is a constant. In fact, letting $t \to \infty$, for $\xi \in \mathbb{R}$,

$$F(z) = A(\xi) = \lim_{t\to\infty} e^{2\pi i\xi t} \int_{-\infty}^{t} f(x)e^{-2\pi i\xi x}\,\mathrm{d}x = 0,$$

since $\hat{f}(\xi) = 0$ and $e^{2\pi i\xi t}$ is bounded.

(c) Hence $F(z) = F(0) = \int_{-\infty}^{t} f(x)\,\mathrm{d}x = 0$ for each t. So for any $\varepsilon > 0$,

$$\int_{t}^{t+\varepsilon} f(x)\,\mathrm{d}x = 0.$$

Consequently, $f(t) = 0$ for each $t \in \mathbb{R}$, since f is continuous.

□

3

PROOF. Consider the integral

$$\int_{\gamma_R} \frac{a}{a^2+z^2} e^{-2\pi iz\xi}\,\mathrm{d}z.$$

Clearly, the function $\frac{a}{a^2+z^2}e^{-2\pi iz\xi}$ has two poles $\pm ia$ with the residues $\pi e^{2\pi a\xi}$ at ia and $-\pi e^{-2\pi a\xi}$. We choose the contour $\gamma_R = [-R, R] \cup C_R^{\pm}$, where C_R^{+} is the large half circle on the upper half plane with counter clockwise direction containing ia and C_R^{-} is the large half circle on the lower half plane with clockwise direction containing $-ia$.

Since $|e^{-2\pi iz\xi}| = e^{2\pi \Im z\xi}$, thus in order to make the function $\frac{a}{a^2+z^2}e^{-2\pi iz\xi}$ to be integrable, it must be that $\Im z\xi < 0$. Thus, when we choose the contour is $\gamma_R = [-R, R] \cup C_R^{+}$, $\xi < 0$, and

$$\int_{-R}^{R} \frac{a}{a^2+x^2} e^{-2\pi ix\xi}\,\mathrm{d}x + \int_{C_R^+} \frac{a}{a^2+z^2} e^{-2\pi iz\xi}\,\mathrm{d}z = 2\pi i \mathrm{Res}_{ia} \frac{a}{a^2+z^2} e^{-2\pi iz\xi} = \pi e^{2\pi a\xi}.$$

Since
$$\int_{C_R^+} \frac{a}{a^2+z^2} e^{-2\pi i z\xi} \,\mathrm{d}z = \int_0^\pi \frac{a}{a^2+R^2e^{2i\theta}} Rie^{i\theta} e^{-2\pi i Re^{i\theta}\xi} \,\mathrm{d}\theta \to 0,$$
as $R \to \infty$ because of $\xi < 0$, thus letting $R \to \infty$,
$$\int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} \,\mathrm{d}x = \pi e^{2\pi a\xi},$$
Similarly, when we choose the contour is $\gamma_R = [-R, R] \cup C_R^-$, $\xi \geq 0$, and
$$\int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} \,\mathrm{d}x = \pi e^{-2\pi a\xi}.$$
Combining these two results,
$$\int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi i x\xi} \,\mathrm{d}x = \pi e^{-2\pi a|\xi|}.$$

□

6

PROOF. From Poisson summation formula,
$$\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{a}{a^2+n^2} = \sum_{n=-\infty}^{\infty} e^{-2\pi a|n|}.$$
From the convergence of power series,
$$\sum_{n=-\infty}^{\infty} e^{-2\pi a|n|} = \coth \pi a.$$

□