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A size-dependent model for bi-layered Kirchhoff micro-plate based on strain gradient elasticity theory

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ABSTRACT

A size-dependent model for bi-layered Kirchhoff micro-plate is developed based on the strain gradient elasticity theory. The governing equations and boundary conditions are derived by using the variational principle. To illustrate the new model, the bending problem of a simply supported bi-layered square micro-plate subjected to constant distributed load is solved. Numerical results reveal that the deflection and axial stress decrease remarkably compared with the classical plate results, and the zero-strain surface deviates significantly from the conventional position, when the thickness of plate is comparable to the material length scale parameters. The size effects, however, are almost diminishing as the thickness of plate is far greater than the material length scale parameters. In addition, the bi-layered plate can be simplified to the monolayer plate as the thickness of one layer is becoming much greater than that of the other layer.

1. Introduction

MEMS (Micro-Electro-Mechanical-System) has been widely used as resonators, biosensors and actuators for its small size, intelligence, conveniently controlling in the field of aerospace, electronics, machinery, medical instruments, civil engineering and so on [1–4]. MEMS devices, according to the geometry and loaded forms, can be simplified to some typical micro-components, such as micro-beam or micro-plate. Since the thicknesses of micro-components are on the order of micron or sub-micro, their mechanical properties are very different from those of macroscopic devices. The mechanical behaviors in micro-structures exhibit obvious size effect, which has been experimentally observed in both metals and polymers [5–7]. The size-dependent behavior cannot be explained by the conventional strain-based theories due to the absence of the internal material length scale parameters. The strain gradient theories have been developed to explain the size dependence of the deformation behavior, in which the material length scale parameters are incorporated into constitutive relations.

According to the deformation metrics used, the strain gradient theories can be classified into couple stress theories and general strain gradient theories. The classical couple stress theory, which uses the higher-order rotation gradients as the deformation metrics, was presented by Mindlin and Tiersten [8] and Toupin [9]. This theory includes two higher-order material constants in addition to the conventional Lame constants. Yang et al. [10], introducing a higher-order equilibrium condition, developed the modified couple stress theory with only one higher-order material parameter. The general strain gradient elasticity theory including five higher-order material constants was firstly proposed by Mindlin [11], in which only the second-order deformation gradients (first-order strain gradients) are included as additional deformation metrics. Also, by using a new set of higher-order metrics and applying the higher-order equilibrium condition, Lam et al. [5] modified the general strain gradient theory and reduced the number of independent higher-order material parameters from five to three. In addition, the simple model with only one additional material constant in the strain gradient elasticity was proposed by Aifantis [12].

In order to explain the size effects in micro-structures, various strain gradient elasticity theories have been used by researchers to develop strain gradient beam and plate theories. For example, the classical couple stress theory has been employed by Anthoine [13] to establish the bending model of a circular cylinder. Park

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and Gao [14] proposed a Bernoulli–Euler beam model based on the modified couple stress theory. The strain gradient elasticity theory has been used by Kong et al. [15] to construct the formulation of a Bernoulli–Euler beam model. By employing the same strain gradient theory, Wang et al. [16] developed a Timoshenko beam model to analyze its static bending and free vibration. For micro-plate, a size-dependent model for the static analysis of Kirchhoff plate with arbitrary shape was presented by Tsiatas [17] based on the modified couple stress theory. Ke et al. [18] and Jomehzadeh et al. [19] employed the same couple stress theory to study the free vibrations of Mindlin micro-plate and Kirchhoff micro-plate, respectively. Based on the strain gradient elasticity theory, a Kirchhoff plate model was developed by Ashoori Movassagh and Mahmoodi [20] and Wang et al. [21].

All above researches are aimed at monolayer micro-components. However, the micro-components are usually bilayered or multilayered structures due to their special micro-machining technology, such as physical and electrochemical depositions [22–24]. Hence, it is essential to develop similar size-dependent models highlighting the laminated micro-components. Zhang et al. [25] studied elastic bending problems of bi-layered micro-cantilever beams subjected to a transverse concentrated load based on the Aifantis' strain gradient elasticity theory. A size-dependent bi-layered microbeam model was developed by Li et al. [26] employing the strain gradient elasticity theory. Researchers further extended the isotropic modified couple stress theory to anisotropic modified couple stress theory and employed this theory to analyze the bending and free vibration of composite laminated beam and plate. Khandan et al. [27] reviewed the development of composite laminated plate theories from very basic classical laminated plate theory to more complicated and higher-order shear deformation theory. The first order shear deformation theory with constant transverse shear stress was proposed by Mindlin [28] and Reissner [29]. Reddy [30] presented a third-order shear deformation theory accounting for parabolic distribution of the transverse shear strains through the thickness of the plate. A model of composite laminated beam based on the global–local theory for new modified couple stress theory was developed by Chen and Si [31]. Roque et al. [32] used the modified couple stress theory to study the bending of simply supported laminated composite Timoshenko beams subjected to transverse loads. The models for composite laminated Reddy beam [33] and plate [34] were developed by Chen et al. employing the modified couple stress theory, respectively. Moreover, for functionally graded beam and plate, Asghari et al. [35], Akgoz and Civalek [36], Reddy and Berry [37], Sahmani and Ansari [38] investigated the static bending and free vibration of FGM micro-beams and micro-plates based on the modified couple stress theory.

In this paper, the bi-layered micro-plate model is developed based on the strain gradient elasticity theory proposed by Lam. The governing equations and boundary conditions are derived by using the variational principle. To illustrate the new model, a boundary value problem of simply supported bi-layered micro-plate is solved. The influences of thicknesses of two layers on the deflection are analyzed. The size effects on deflection, axial stress and location of zero-strain surface are discussed.

2. Size-dependent bi-layered Kirchhoff micro-plate model

2.1. Strain gradient elasticity theory

Lam et al. [5] developed a strain gradient elasticity theory with three independent material length scale parameters. In this theory, the dilatation gradient tensor γ_i, the deviatoric stretch gradient tensor $\eta_{ijk}^{(1)}$ and the symmetric rotation gradient tensor χ_{ij}^s are introduced except the classical strain tensor ε_{ij}. These deformation measures are defined as

$$\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i), \tag{1}$$

$$\gamma_i = \partial_i \varepsilon_{mm}, \tag{2}$$

$$\eta_{ijk}^{(1)} = \frac{1}{3}(\partial_i \varepsilon_{jk} + \partial_j \varepsilon_{ki} + \partial_k \varepsilon_{ij}) - \frac{1}{15}[\delta_{ij}(\partial_k \varepsilon_{mm} + 2\partial_m \varepsilon_{mk}) + \delta_{jk}(\partial_i \varepsilon_{mm} + 2\partial_m \varepsilon_{mi}) + \delta_{ki}(\partial_j \varepsilon_{mm} + 2\partial_m \varepsilon_{mj})], \tag{3}$$

$$\chi_{ij}^s = \frac{1}{2}(e_{ipq}\partial_p \varepsilon_{qj} + e_{jpq}\partial_p \varepsilon_{qi}), \tag{4}$$

where u_i is the displacement vector, ∂_i is the differential operator, ε_{mm} is the dilatation strain, δ_{ij} is the Kronecker symbol and e_{ijk} is the alternate symbol.

For the isotropic linear elastic material, the strain energy density w_0 is given as

$$w_0 = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + \mu l_0^2 \gamma_i \gamma_i + \mu l_1^2 \eta_{ijk}^{(1)} \eta_{ijk}^{(1)} + \mu l_2^2 \chi_{ij}^s \chi_{ij}^s, \tag{5}$$

where λ and μ are the Lame constants, l_0, l_1 and l_2 are the independent material length scale parameters associated with the dilation gradients, deviatoric stretch gradients and symmetric rotation gradients, respectively.

2.2. Governing equation and boundary conditions

Consider a bi-layered rectangular elastic micro-plate subjected to a static transverse load $q(x,y)$ distributed in the x–y plane as shown in Fig. 1. The length and width of the plate are a, b, and the thicknesses of the lower and upper layers are h_1 and h_2, respectively. The properties of materials are $E_{(1)}$, $\upsilon_{(1)}$, $l_{0(1)}$, $l_{1(1)}$, $l_{2(1)}$ and $E_{(2)}$, $\upsilon_{(2)}$, $l_{0(2)}$, $l_{1(2)}$, $l_{2(2)}$, where E is the Young's modulus, υ is the poisson's ratio and subscripts 1 and 2 in brackets denote the lower and upper layers, respectively. The position of neutral surface is assumed to be deviated d from the interface between two layers.

For the Kirchhoff plate, where x_0–y_0 plane is coincident with the neutral surface, the displacement components are taking as

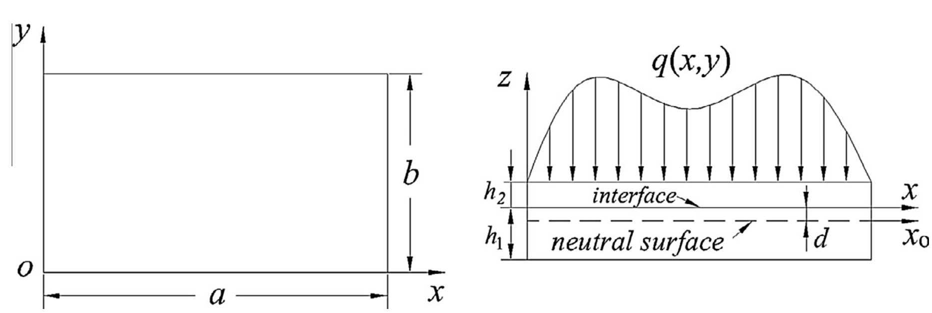

Fig. 1. Schematic of a bilayered micro-plate with distributed load.

$$
\begin{aligned}
u(x,y,z) &= -z\frac{\partial w(x,y)}{\partial x} \\
v(x,y,z) &= -z\frac{\partial w(x,y)}{\partial y} \\
w(x,y,z) &= w(x,y)
\end{aligned} \tag{6}
$$

in which u, v and w are displacement components along x, y, z directions. When the x_0–y_0 plane is translated d from the neutral surface to the interface, the displacement components of plate can be expressed as

$$
\begin{aligned}
u(x,y,z) &= -(d+z)\frac{\partial w(x,y)}{\partial x}, \\
v(x,y,z) &= -(d+z)\frac{\partial w(x,y)}{\partial y}, \quad w(x,y,z) = w(x,y).
\end{aligned} \tag{7}
$$

This displacement field can be considered as the displacement field of bi-layered plates when the location of neutral surface cannot be known in advance. In this case, the x–y coordinate plane can be established at the interface. Considering the in-plane displacement at the interface, $u_0 = -d \cdot \partial w/\partial x$, $v_0 = -d \cdot \partial w/\partial y$, the displacement field of plate Eq. (7) can also be written as

$$
\begin{aligned}
u(x,y,z) &= u_0(x,y) - z\frac{\partial w(x,y)}{\partial x} \\
v(x,y,z) &= v_0(x,y) - z\frac{\partial w(x,y)}{\partial y} \\
w(x,y,z) &= w(x,y).
\end{aligned} \tag{8}
$$

The non-zero strain components obtained from Eqs. (1) and (8) are written as

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_{yy} = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w}{\partial y^2}, \quad \varepsilon_{xy} = \varepsilon_{yx} \\
&= \frac{1}{2}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y}\right).
\end{aligned} \tag{9}
$$

By substituting Eq. (9) into Eqs. (2)–(4), the strain gradient tensors γ_i, $\eta^{(1)}_{ijk}$, χ^s_{ij} are obtained and presented in Appendix A. Subsequently, the strain and strain gradient tensors can be substituted into Eq. (5), and then the strain energy density of monolayer plate is calculated and expressed as

$$
\begin{aligned}
w_0 = {} & (c_1 + c_2 z^2)\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] \\
& + c_2\left[\left(\frac{\partial u_0}{\partial x}\right)^2 + \left(\frac{\partial v_0}{\partial y}\right)^2 - 2z\left(\frac{\partial u_0}{\partial x}\frac{\partial^2 w}{\partial x^2} + \frac{\partial v_0}{\partial y}\frac{\partial^2 w}{\partial y^2}\right)\right] \\
& + (c_3 + c_4 z^2)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \\
& + c_4\left[\frac{\partial u_0}{\partial x}\frac{\partial v_0}{\partial y} - z\left(\frac{\partial u_0}{\partial x}\frac{\partial^2 w}{\partial y^2} + \frac{\partial v_0}{\partial y}\frac{\partial^2 w}{\partial x^2}\right)\right] \\
& + (c_5 + c_6 z^2)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \\
& + \frac{1}{4}c_6\left[\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right)^2 - 4z\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right)\frac{\partial^2 w}{\partial x \partial y}\right] \\
& + c_7\left[\left(\frac{\partial^2 u_0}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v_0}{\partial y^2}\right)^2\right] \\
& - c_7\left[2z\left(\frac{\partial^2 u_0}{\partial x^2}\frac{\partial^3 w}{\partial x^3} + \frac{\partial^2 v_0}{\partial y^2}\frac{\partial^3 w}{\partial y^3}\right) - z^2\left(\left(\frac{\partial^3 w}{\partial x^3}\right)^2 + \left(\frac{\partial^3 w}{\partial y^3}\right)^2\right)\right] \\
& + c_8 z^2\left[\left(\frac{\partial^3 w}{\partial x \partial y^2}\right)^2 + \left(\frac{\partial^3 w}{\partial y \partial x^2}\right)^2\right] \\
& + c_9\left[z^2\left(\frac{\partial^3 w}{\partial x^3}\frac{\partial^3 w}{\partial x \partial y^2} + \frac{\partial^3 w}{\partial y^3}\frac{\partial^3 w}{\partial y \partial x^2}\right)\right. \\
& \left. - z\left(\frac{\partial^2 u_0}{\partial x^2}\frac{\partial^3 w}{\partial x \partial y^2} + \frac{\partial^2 v_0}{\partial y^2}\frac{\partial^3 w}{\partial x^2 \partial y}\right)\right] \\
& + c_{10}\left[\left(\frac{\partial^2 u_0}{\partial y^2}\right)^2 + \left(\frac{\partial^2 v_0}{\partial x^2}\right)^2\right] + c_{11}\left(\frac{\partial^2 u_0}{\partial y^2}\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial x^2}\frac{\partial^2 v_0}{\partial y^2}\right) \\
& - c_{12} z\left(\frac{\partial^2 v_0}{\partial x \partial y}\frac{\partial^3 w}{\partial x \partial y^2} + \frac{\partial^2 u_0}{\partial x \partial y}\frac{\partial^3 w}{\partial x^2 \partial y}\right) \\
& - c_{11} z\left(\frac{\partial^2 u_0}{\partial y^2}\frac{\partial^3 w}{\partial x^3} + \frac{\partial^2 v_0}{\partial x^2}\frac{\partial^3 w}{\partial y^3} - 4\frac{\partial^2 u_0}{\partial y^2}\frac{\partial^3 w}{\partial x \partial y^2} - 4\frac{\partial^2 v_0}{\partial x^2}\frac{\partial^3 w}{\partial x^2 \partial y}\right) \\
& + c_{13}\left[\left(\frac{\partial^2 u_0}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 v_0}{\partial x \partial y}\right)^2\right] \\
& + c_{14}\left[\frac{\partial^2 u_0}{\partial x^2}\frac{\partial^2 v_0}{\partial x \partial y} + \frac{\partial^2 v_0}{\partial y^2}\frac{\partial^2 u_0}{\partial x \partial y} - z\left(\frac{\partial^3 w}{\partial x^3}\frac{\partial^2 v_0}{\partial x \partial y} + \frac{\partial^3 w}{\partial y^3}\frac{\partial^2 u_0}{\partial x \partial y}\right)\right] \\
& + c_{15}\left(\frac{\partial^2 u_0}{\partial y^2}\frac{\partial^2 v_0}{\partial x \partial y} + \frac{\partial^2 v_0}{\partial x^2}\frac{\partial^2 u_0}{\partial x \partial y}\right)
\end{aligned} \tag{10}
$$

in which the parameters c_i ($i = 1, 2, 3 \ldots, 15$) are given as

$$
\begin{aligned}
& c_1 = \mu l_0^2 + \frac{4}{15}\mu l_1^2 + \frac{1}{2}\mu l_2^2, \quad c_2 = \frac{1}{2}\frac{E}{1-v^2}, \\
& c_3 = 2\mu l_0^2 - \frac{2}{15}\mu l_1^2 - \mu l_2^2 \\
& c_4 = \frac{Ev}{1-v^2}, \quad c_5 = \frac{2}{3}\mu l_1^2 + 2\mu l_2^2, \quad c_6 = 2\mu, \quad c_7 = \mu l_0^2 + \frac{2}{5}\mu l_1^2, \\
& c_8 = \mu l_0^2 + \frac{12}{5}\mu l_1^2 \\
& c_9 = 2\mu l_0^2 - \frac{6}{5}\mu l_1^2, \quad c_{10} = \frac{4}{15}\mu l_1^2 + \frac{1}{8}\mu l_2^2, \quad c_{11} = -\frac{2}{5}\mu l_1^2, \\
& c_{12} = \frac{16}{5}\mu l_1^2 + 2\mu l_0^2 \\
& c_{13} = \mu l_0^2 + \frac{16}{15}\mu l_1^2 + \frac{1}{8}\mu l_2^2, \quad c_{14} = 2\mu l_0^2 - \frac{4}{5}\mu l_1^2, \\
& c_{15} = \frac{16}{15}\mu l_1^2 - \frac{1}{4}\mu l_2^2
\end{aligned} \tag{11}
$$

For the bi-layered plate composed of two different materials, the parameters $c_{i(1)}$ and $c_{i(2)}$ ($i = 1, 2, 3 \ldots, 15$) associated with the lower and upper layers can be obtained by substituting $E_{(1)}$, $v_{(1)}$, $\mu_{(1)}$, $l_{0(1)}$, $l_{1(1)}$, $l_{2(1)}$ and $E_{(2)}$, $v_{(2)}$, $\mu_{(2)}$, $l_{0(2)}$, $l_{1(2)}$, $l_{2(2)}$ for E, v, μ, l_0, l_1, l_2 in Eq. (11), respectively. Moreover, the strain energy densities of the lower and upper layers, $w_{0(1)}$ and $w_{0(2)}$, can be obtained by substituting $c_{i(1)}$ and $c_{i(2)}$ for c_i ($i = 1, 2, 3 \ldots, 15$) in Eq. (10), respectively.

The total strain energy can be obtained by integrating strain energy densities on the total plate. It can be written as

$$
\begin{aligned}
U &= \int\int_A \left(\int_{-h_1}^{0} w_{0(1)} dz\right) dxdy + \int\int_A \left(\int_0^{h_2} w_{0(2)} dz\right) dxdy \\
&= \int\int_A F dxdy,
\end{aligned} \tag{12}
$$

in which the specific form of lagrange function F is presented in Appendix A. The total work done by the external distributed force $q(x,y)$ are given as

$$
W = \int_0^a \int_0^b q(x,y) w(x,y) dxdy. \tag{13}
$$

The governing equation as well as boundary conditions of a plate in bending can be derived by using the principle of minimum potential energy.

$$\begin{aligned}
\delta(U-W)=0=&\int_A\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial w_{xx}}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial w_{yy}}\right)+\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial F}{\partial w_{xy}}\right)\right.\\
&-\frac{\partial^3}{\partial x^3}\left(\frac{\partial F}{\partial w_{xxx}}\right)-\frac{\partial^3}{\partial y^3}\left(\frac{\partial F}{\partial w_{yyy}}\right)-\frac{\partial^3}{\partial x\partial y^2}\left(\frac{\partial F}{\partial w_{xyy}}\right)\\
&\left.-\frac{\partial^3}{\partial y\partial x^2}\left(\frac{\partial F}{\partial w_{xxy}}\right)-q\right]\delta w dxdy\\
&+\int_0^a\left[\left(\frac{\partial^2}{\partial y^2}\frac{\partial F}{\partial w_{yyy}}-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial w_{yy}}\right)-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial w_{xy}}\right)\right.\right.\\
&\left.+\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial F}{\partial w_{xyy}}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial w_{xxy}}\right)\right)\delta w|_0^b\\
&+\left(\frac{\partial F}{\partial w_{yy}}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial w_{xyy}}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial w_{yyy}}\right)\right)\delta w_y|_0^b\\
&\left.+\frac{\partial F}{\partial w_{yyy}}\delta w_{yy}|_0^b\right]dx+\int_0^b\left[\left(\frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial w_{xyy}}\right)+\frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial w_{xxx}}\right.\right.\\
&\left.+\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial F}{\partial w_{xxy}}\right)-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial w_{xx}}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial w_{xy}}\right)\right)\delta w|_0^a\\
&+\left(\frac{\partial F}{\partial w_{xx}}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial w_{xxx}}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial w_{xxy}}\right)\right)\delta w_x|_0^a\\
&\left.+\frac{\partial F}{\partial w_{xxx}}\delta w_{xx}|_0^a\right]dy+\int_A\left[-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_{0x}}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_{0y}}\right)\right.\\
&\left.+\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial u_{0xx}}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial u_{0yy}}\right)+\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial F}{\partial u_{0xy}}\right)\right]\delta u_0\\
&+\left[-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_{0x}}\right)-\frac{\partial}{\partial y}(\frac{\partial F}{\partial v_{0y}})+\frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial v_{0xx}}\right)\right.\\
&\left.+\frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial v_{0yy}}\right)+\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial F}{\partial v_{0xy}}\right)\right]\delta v_0 dxdy\\
&+\int_0^a\left[\left(\frac{\partial F}{\partial u_{0y}}-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_{0yy}}\right)-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_{0xy}}\right)\right)\delta u_0\right.\\
&+\left(\frac{\partial F}{\partial v_{0y}}-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_{0yy}}\right)-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_{0xy}}\right)\right)\delta v_0\\
&\left.+\frac{\partial F}{\partial u_{0yy}}\delta u_{0y}+\frac{\partial F}{\partial v_{0yy}}\delta v_{0y}\right]|_0^b dx\\
&+\int_0^b\left[\left(\frac{\partial F}{\partial u_{0x}}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_{0xx}}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_{0xy}}\right)\right)\delta u_0\right.\\
&+\left(\frac{\partial F}{\partial v_{0x}}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_{0xx}}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_{0xy}}\right)\right)\delta v_0\\
&\left.+\frac{\partial F}{\partial u_{0xx}}\delta u_{0x}+\frac{\partial F}{\partial v_{0xx}}\delta v_{0x}\right]|_0^a dy
\end{aligned}\tag{14}$$

in which

$$\begin{aligned}
&w_{xx}=\frac{\partial^2 w}{\partial x^2},\quad w_{yy}=\frac{\partial^2 w}{\partial y^2},\quad w_{xy}=\frac{\partial^2 w}{\partial x\partial y},\quad w_{xxx}=\frac{\partial^3 w}{\partial x^3},\\
&w_{yyy}=\frac{\partial^3 w}{\partial y^3},\quad w_{xxy}=\frac{\partial^3 w}{\partial x^2\partial y}\\
&w_{xyy}=\frac{\partial^3 w}{\partial x\partial y^2},\quad u_{0x}=\frac{\partial u_0}{\partial x},\quad u_{0y}=\frac{\partial u_0}{\partial y},\quad u_{0xx}=\frac{\partial^2 u_0}{\partial xx},\\
&u_{0yy}=\frac{\partial^2 u_0}{\partial y^2},\quad u_{0xy}=\frac{\partial^2 u_0}{\partial x\partial y}\\
&v_{0x}=\frac{\partial v_0}{\partial x},\quad v_{0y}=\frac{\partial v_0}{\partial y},\quad v_{0xx}=\frac{\partial^2 v_0}{\partial xx},\quad v_{0yy}=\frac{\partial^2 v_0}{\partial y^2},\\
&v_{0xy}=\frac{\partial^2 v_0}{\partial x\partial y}
\end{aligned}\tag{15}$$

Substituting the lagrange function F into Eq. (14), the governing equations are given by

$$\begin{aligned}
&-2a_{12}\nabla^6 w+2a_1\nabla^4 w+2a_{11}\left(\frac{\partial^5 u_0}{\partial x^5}+2\frac{\partial^5 u_0}{\partial x^3\partial y^2}+\frac{\partial^5 u_0}{\partial x\partial y^4}+\frac{\partial^5 v_0}{\partial x^4\partial y}\right.\\
&\left.+2\frac{\partial^5 v_0}{\partial x^2\partial y^3}+\frac{\partial^5 v_0}{\partial y^5}\right)-2a_3\left(\frac{\partial^3 u_0}{\partial x^3}+\frac{\partial^3 u_0}{\partial x\partial y^2}+\frac{\partial^3 v_0}{\partial x^2\partial y}+\frac{\partial^3 v_0}{\partial y^3}\right)=q
\end{aligned}\tag{16}$$

$$\begin{aligned}
&-2a_{11}\left(\frac{\partial^5 w}{\partial x^5}+2\frac{\partial^5 w}{\partial x^3\partial y^2}+\frac{\partial^5 w}{\partial x\partial y^4}\right)+2a_3\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x\partial y^2}\right)\\
&+2a_{10}\frac{\partial^4 u_0}{\partial x^4}+2(a_{17}+a_{20})\frac{\partial^4 u_0}{\partial x^2\partial y^2}+(a_{21}+a_{23})\left(\frac{\partial^4 v_0}{\partial x^3\partial y}+\frac{\partial^4 v_0}{\partial x\partial y^3}\right)\\
&+2a_{16}\frac{\partial^4 u_0}{\partial y^4}-2a_2\frac{\partial^2 u_0}{\partial x^2}-(a_5+2a_8)\frac{\partial^2 v_0}{\partial x\partial y}-2a_8\frac{\partial^2 u_0}{\partial y^2}=0
\end{aligned}\tag{17}$$

$$\begin{aligned}
&-2a_{11}\left(\frac{\partial^5 w}{\partial y^5}+2\frac{\partial^5 w}{\partial x^2\partial y^3}+\frac{\partial^5 w}{\partial x^4\partial y}\right)+2a_3\left(\frac{\partial^3 w}{\partial y^3}+\frac{\partial^3 w}{\partial x^2\partial y}\right)\\
&+2a_{10}\frac{\partial^4 v_0}{\partial y^4}+2(a_{17}+a_{20})\frac{\partial^4 v_0}{\partial x^2\partial y^2}+(a_{21}+a_{23})\left(\frac{\partial^4 u_0}{\partial x^3\partial y}+\frac{\partial^4 u_0}{\partial x\partial y^3}\right)\\
&+2a_{16}\frac{\partial^4 v_0}{\partial x^4}-2a_2\frac{\partial^2 v_0}{\partial y^2}-(a_5+2a_8)\frac{\partial^2 u_0}{\partial x\partial y}-2a_8\frac{\partial^2 v_0}{\partial x^2}=0
\end{aligned}\tag{18}$$

in which

$$\begin{aligned}
\nabla^4&=\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\\
\nabla^6&=\frac{\partial^6}{\partial x^6}+3\frac{\partial^6}{\partial x^4\partial y^2}+3\frac{\partial^6}{\partial x^2\partial y^4}+\frac{\partial^6}{\partial y^6}
\end{aligned}\tag{19}$$

The boundary conditions are written as

$$\begin{aligned}
&B_{X1}(a,y)\delta w(a,y)-B_{X1}(0,y)\delta w(0,y)=0,\\
&B_{Y1}(x,b)\delta w(x,b)-B_{Y1}(x,0)\delta w(x,0)=0\\
&B_{X2}(a,y)\delta w_x(a,y)-B_{X2}(0,y)\delta w_x(0,y)=0,\\
&B_{Y2}(x,b)\delta w_y(x,b)-B_{Y2}(x,0)\delta w_y(x,0)=0\\
&B_{X3}(a,y)\delta w_{xx}(a,y)-B_{X3}(0,y)\delta w_{xx}(0,y)=0,\\
&B_{Y3}(x,b)\delta w_{yy}(x,b)-B_{Y3}(x,0)\delta w_{yy}(x,0)=0\\
&B_{X4}(a,y)\delta u_0(a,y)-B_{X4}(0,y)\delta u_0(0,y)=0,\\
&B_{Y4}(x,b)\delta v_0(x,b)-B_{Y4}(x,0)\delta v_0(x,0)=0\\
&B_{X5}(a,y)\delta v_0(a,y)-B_{X5}(0,y)\delta v_0(0,y)=0,\\
&B_{Y5}(x,b)\delta u_0(x,b)-B_{Y5}(x,0)\delta u_0(x,0)=0\\
&B_{X6}(a,y)\delta v_{0x}(a,y)-B_{X6}(0,y)\delta v_{0x}(0,y)=0,\\
&B_{Y6}(x,b)\delta u_{0y}(x,b)-B_{Y6}(x,0)\delta u_{0y}(x,0)=0\\
&B_{X7}(a,y)\delta u_{0x}(a,y)-B_{X7}(0,y)\delta u_{0x}(0,y)=0,\\
&B_{Y7}(x,b)\delta v_{0y}(x,b)-B_{Y7}(x,0)\delta v_{0y}(x,0)=0
\end{aligned}\tag{20}$$

in which the specific forms of $B_{Xi}(x,y)$ and $B_{Yi}(x,y)$ (i = 1, 2, ..., 7) are shown in Appendix A. When the materials and thicknesses of the lower and upper layers are the same, the governing equations and boundary conditions Eqs. (16)–(20) reduce to that of the monolayer plate [20,21].

3. Static bending of simply supported bi-layered Kirchhoff micro-plate

3.1. Solutions of a simply supported bi-layered Kirchhoff micro-plate

For simply supported plate, deflection w is constant and identical to zero at all edges. Hence, the slope of w in the y direction, $\partial w/$

∂y, along the edges $x = 0$, a, and the slope of w in the x direction, $\partial w/\partial x$, along the edges $y = 0$, b are equal to zero. Considering $u_0 = -d \cdot \partial w/\partial x$, $v_0 = -d \cdot \partial w/\partial y$, the displacement v_0 along the edges $x = 0$, a, and the displacement u_0 along the edges $y = 0$, b, are also constant and zero. The slope of v_0 in the y direction, $\partial v_0/\partial y$, along the edges $x = 0$, a, and the slope of u_0 in the x direction, $\partial u_0/\partial x$, along the edges $y = 0$, b, are also equal to zero. Consequently, the following equalities can be obtained.

$$\begin{aligned} &x=0, \quad a \; w=0, \quad \frac{\partial w}{\partial y}=0, \quad v_0=0, \quad \frac{\partial v_0}{\partial y}=0 \\ &y=0, \; b \quad w=0, \quad \frac{\partial w}{\partial x}=0, \quad u_0=0, \quad \frac{\partial u_0}{\partial x}=0. \end{aligned} \tag{21}$$

Based on Eq. (21), the boundary conditions Eq. (20) can be simplified. From Eq. (20), the first and fifth equations are simplified as

$$\begin{aligned} &w(0,y)=w(a,y)=0, \quad w(x,0)=w(x,b)=0 \\ &v_0(0,y)=v_0(a,y)=0, \quad u_0(x,0)=u_0(x,b)=0, \end{aligned} \tag{22}$$

and $B_{Xi}(x,y)$ and $B_{Yi}(x,y)$ ($i = 2, 3, 4, 6, 7$) in the remaining five equations are simplified to $B_{xi}(x,y)$ and $B_{yi}(x,y)$ ($i = 2, 3, 4, 6, 7$), which have been presented in Appendix A. Eventually, the boundary conditions are given as

$$\begin{aligned} &w(0,y)=w(a,y)=0, \quad w(x,0)=w(x,b)=0 \\ &v_0(0,y)=v_0(a,y)=0, \quad u_0(x,0)=u_0(x,b)=0 \\ &w_{xx}(0,y)=w_{xx}(a,y)=w_{yy}(x,0)=w_{yy}(x,b)=0 \\ &u_{0x}(0,y)=u_{0x}(a,y)=v_{0y}(x,0)=v_{0y}(x,b)=0 \\ &B_{x2}(a,y)=B_{x2}(0,y)=B_{y2}(x,b)=B_{y2}(x,0)=0 \\ &B_{x4}(a,y)=B_{x4}(0,y)=B_{y4}(x,b)=B_{y4}(x,0)=0 \\ &B_{x6}(a,y)=B_{x6}(0,y)=B_{y6}(x,b)=B_{y6}(x,0)=0. \end{aligned} \tag{23}$$

To solve the governing Eqs. (16)–(18) under the boundary conditions Eq. (23), the following Fourier series is assumed.

$$\begin{aligned} &w(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \\ &u_0(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \\ &v_0(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \end{aligned} \tag{24}$$

where A_{mn}, B_{mn}, C_{mn} are the Fourier coefficients to be determined by m and n. It is obvious that Eq. (24) satisfies all the boundary conditions in Eq. (23). And the distributed transverse load $q(x,y)$ can also be expressed as Fourier series:

$$q(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Q_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \tag{25}$$

For the uniformly distributed load $q(x,y) = q_0$, Q_{mn} is expressed as [39]

$$Q_{mn}=\frac{16q_0}{mn\pi^2} \quad m,n=1,3,5,\ldots \tag{26}$$

Substituting Eqs. (24) and (25) into Eqs. (16)–(18), Fourier coefficients can be deduced as

$$\begin{aligned} &A_{mn}=Q_{mn}\frac{P_6^2-P_4P_5}{P_3^2P_4-P_1P_4P_5-2P_2P_3P_6+P_1P_6^2+P_2^2P_5} \\ &B_{mn}=Q_{mn}\frac{P_3P_6-P_2P_5}{P_3^2P_4-P_1P_4P_5-2P_2P_3P_6+P_1P_6^2+P_2^2P_5} \\ &C_{mn}=Q_{mn}\frac{P_2P_6-P_3P_4}{P_3^2P_4-P_1P_4P_5-2P_2P_3P_6+P_1P_6^2+P_2^2P_5} \end{aligned} \tag{27}$$

in which

$$\begin{aligned} P_1 &= 2a_{12}\left(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right)^3+2a_1\left(\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right)^2 \\ P_2 &= 2a_{11}\left(\left(\frac{m\pi}{a}\right)^5+2\left(\frac{m\pi}{a}\right)^3\left(\frac{n\pi}{b}\right)^2+\frac{m\pi}{a}\left(\frac{n\pi}{b}\right)^4\right) \\ &\quad +2a_3\left(\left(\frac{m\pi}{a}\right)^3+\frac{m\pi}{a}\left(\frac{n\pi}{b}\right)^2\right) \\ P_3 &= 2a_{11}\left(\left(\frac{n\pi}{b}\right)^5+2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^3+\left(\frac{m\pi}{a}\right)^4\frac{n\pi}{b}\right) \\ &\quad +2a_3\left(\left(\frac{n\pi}{b}\right)^3+\left(\frac{m\pi}{a}\right)^2\frac{n\pi}{b}\right) \\ P_4 &= 2a_{10}\left(\frac{m\pi}{a}\right)^4+2a_{16}\left(\frac{n\pi}{b}\right)^4+2(a_{17}+a_{20})\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 \\ &\quad +2a_2\left(\frac{m\pi}{a}\right)^2+2a_8\left(\frac{n\pi}{b}\right)^2 \\ P_5 &= 2a_{10}\left(\frac{n\pi}{b}\right)^4+2a_{16}\left(\frac{m\pi}{a}\right)^4+2(a_{17}+a_{20})\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 \\ &\quad +2a_2\left(\frac{n\pi}{b}\right)^2+2a_8\left(\frac{m\pi}{a}\right)^2 \\ P_6 &= (a_{21}+a_{23})\left(\left(\frac{m\pi}{a}\right)^3\frac{n\pi}{b}+\frac{m\pi}{a}\left(\frac{n\pi}{b}\right)^3\right)+(a_5+2a_8)\frac{m\pi}{a}\frac{n\pi}{b} \end{aligned} \tag{28}$$

3.2. Numerical results

For simplification, three material length scale parameters are taken the same for the same materials, $l_{0(1)} = l_{1(1)} = l_{2(1)} = l_{(1)}$, $l_{0(2)} = l_{1(2)} = l_{2(2)} = l_{(2)}$. To illustrate the solutions of the simply supported bi-layered micro-plate, some numerical results are shown under the conditions of $E_{(1)} = 130$ GPa, $E_{(2)} = 85$ GPa, $\nu = 0.3$, $l_{(1)} = 2l_{(2)} = 2l$, $a = b = 30$ μm, $q = 10$ μN. Enough precise results can be achieved with $m = 31$ and $n = 31$ in the calculation. Consequently, for this square plate, the x- and y-directions are relative and interchangeable. The maximum deflection appears at the center point ($x = 15$ and $y = 15$) and the deflection on the center sections ($x = 15$ or $y = 15$) is the greatest among all the sections.

In order to reveal the effect of thickness ratio of the upper layer to the lower layer, $e = h_2/h_1$, on the deflection, the dimensionless relative thickness t is introduced

$$t=\frac{h_1-h_2}{\sqrt{h_1h_2}}=\frac{e-1}{\sqrt{e}}. \tag{29}$$

Assuming the total thickness of the bi-layered plate, $h = h_1 + h_2$, equals 2.4 μm, the deflection of the plate at central point varying with dimensionless relative thickness is shown in Fig. 2. It reveals that the deflection increases as the dimensionless relative thickness increases. When the thickness of one layer is much greater than that of the other layer, the deflection of the bi-layered plate approaches to that of the monolayer plate.

By using $e = 0.2$, the deflection on the center section of $x = 15$ μm is shown in Fig. 3, and the normalized deflection at central point w/w_c, where w_c is the deflection in the classical theory, is shown in Fig. 4. From Figs. 3 and 4, it can be seen that the deflection decreases significantly in comparison with the classical one when the plate thickness is close to the material length scale parameters. But the size effect is almost diminishing as the thickness of the plate is far greater than the material length scale parameters.

The distribution of axial stress at central point, σ_{xx}, is shown in Fig. 5. From Fig. 5, it can be found that the axial stress is piecewise linear and it jumps at the interface between the upper and lower layers because of their differences of material properties. On the other hand, the total axial stress decreases significantly in comparison with the classical one when the beam thickness is close to the

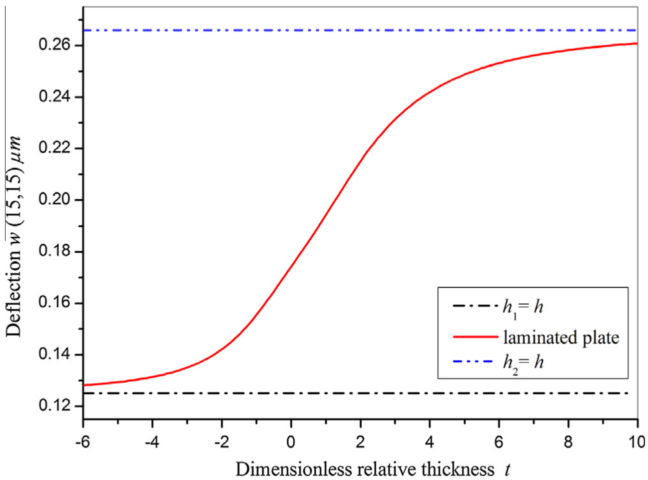

Fig. 2. The deflection at central point varying with dimensionless relative thickness.

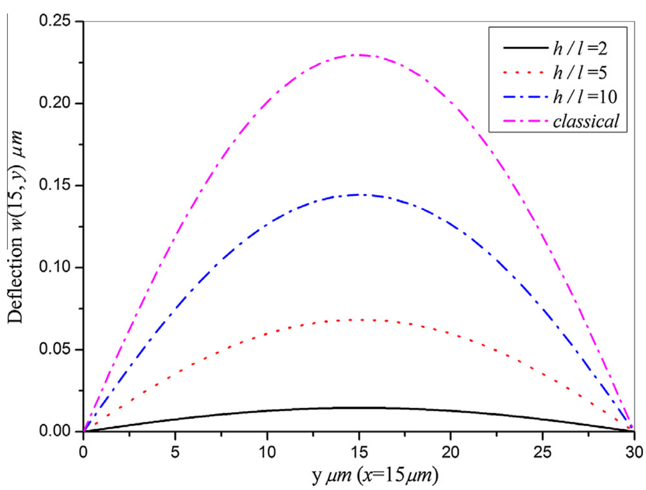

Fig. 3. The deflection distribution on the center section of $x = 15$ μm.

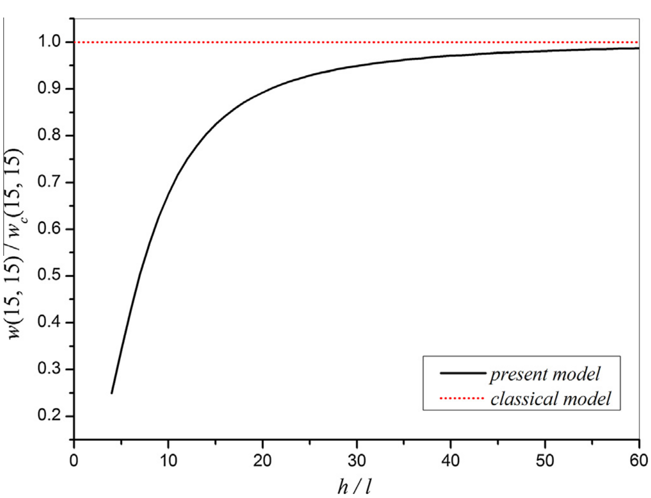

Fig. 4. Size effect on the deflection.

material length scale parameters. But the axial stress is close to the classical one as the thickness of the beam is far greater than the material length scale parameters.

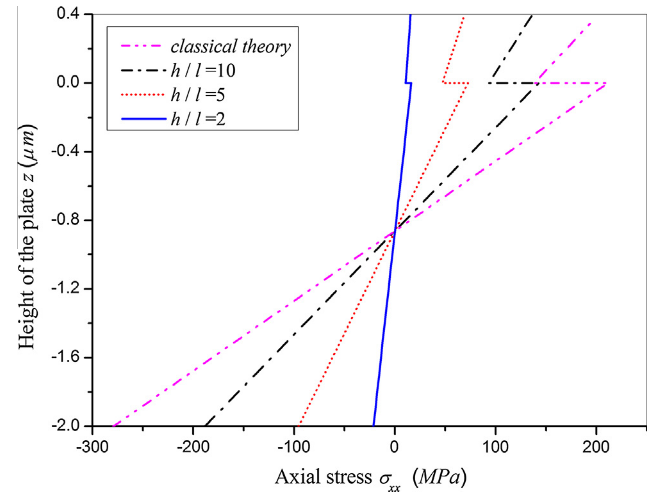

Fig. 5. The distribution of axial stress σ_{xx} along the thickness of plate at central point.

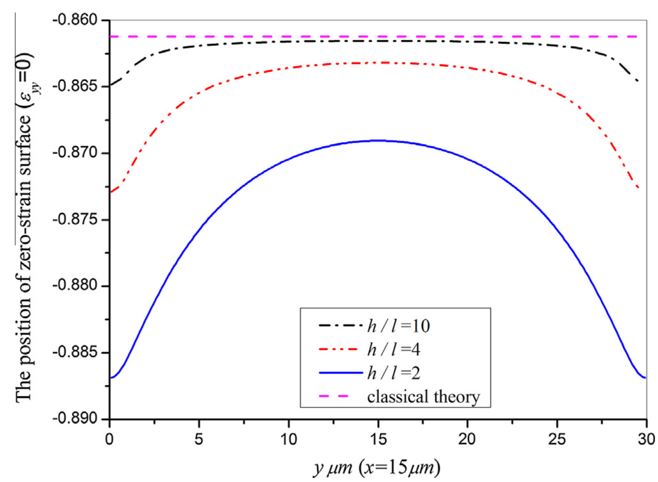

Fig. 6. The location of zero-strain surface on the center section of $x = 15$ μm.

Based on $\varepsilon_{yy} = 0$, the location of zero-strain surface, on the center section of $x = 15$ μm, varying with coordinate y is shown in Fig. 6. From the curves in Fig. 6, it can be observed that the location of zero-strain surface predicted by the classical theory remains unchanged along the coordinate y. While based on the present strain gradient elasticity theory, the location of zero-strain surface moves along the coordinate y. Furthermore, when the thickness of plate is comparable to the material length scale parameters, the zero-strain surface deviates remarkably from that predicted by the classical theory. It gradually approaches to that predicted by the classical theory as the thickness of plate is far greater than the material length scale parameters. In brief, the location of zero-strain surface exhibits obvious size effect.

Fig. 7 shows the distribution of axial strain, ε_{yy}, near the zero-strain surface on the center section of $x = 15$ μm. For the present boundary conditions, $v_{0y}(x,0) = v_{0y}(x,b) = 0$, $w_{yy}(x,0) = w_{yy}(x,b) = 0$, the axial strains equal zero at the both ends and it can be clearly seen in the present figure. For the bi-layered strain gradient micro-plate, the gradients of axial strain are not equal to zero and the zero-strain surface is a curved surface. Moreover, due to the differences of material properties of the lower and upper layers, especially the differences of material length scale parameters, there is three extreme values of axial strains near the zero-strain surface. Accordingly, the axial strains at a plane do not equal zero

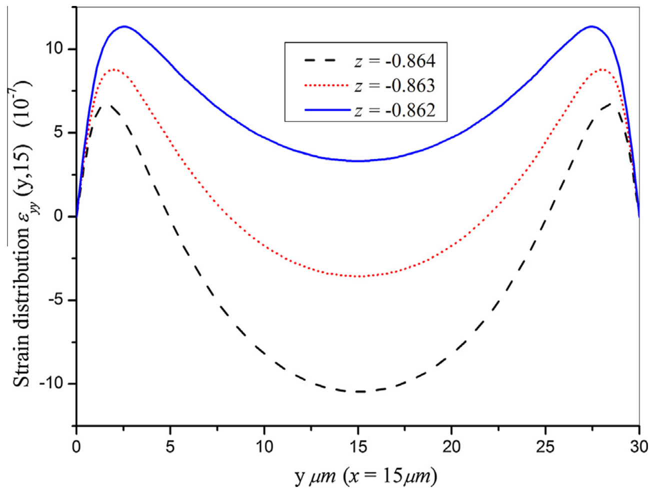

Fig. 7. Distribution of axial strain ε_{yy} near the zero-strain surface on the center section of x = 15 μm.

at the same time. However, for the monolayer strain gradient micro-plate and bi-layered conventional plate, although the gradients of axial strain do not equal zero, there is only one extreme value of axial strain. Thus, the zero-strain surface is a plane.

4. Conclusion

In this paper, the strain gradient elasticity theory presented by Lam is employed to establish an analytical model for the elastic bending problem of a bi-layered micro-plate. The governing equations and boundary conditions are derived by using the variational principle. This new model can be degenerated to that in the classical theory when the high order material constants equal zero. A simply supported bi-layered square micro-plate subjected to a constant distributed load is solved and some numerical results are presented. The results show that the deflection and axial stress of plate and locations of zero-strain surface exhibit obvious size effect. And the results of bi-layered plate approach to that of monolayer plate as the thickness of one layer is becoming much greater than that of the other layer.

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Appendix A

The dilatation gradient tensor, γ_i (i = 1, 2, 3), is

$$\gamma_x = \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial x \partial y} - z\frac{\partial^3 w}{\partial x^3} - z\frac{\partial^3 w}{\partial x \partial y^2}$$
$$\gamma_y = \frac{\partial^2 v_0}{\partial y^2} + \frac{\partial^2 u_0}{\partial x \partial y} - z\frac{\partial^3 w}{\partial y^3} - z\frac{\partial^3 w}{\partial y \partial x^2} \qquad \text{(A.1)}$$
$$\gamma_z = -\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right),$$

The deviatoric stretch gradient tensor, $\eta^{(1)}_{ijk}$ $(i = 1, 2, 3)$, is

$$\eta^{(1)}_{111} = \frac{2}{5}\frac{\partial^2 u_0}{\partial x^2} - \frac{2}{5}\frac{\partial^2 v_0}{\partial x \partial y} - \frac{1}{5}\frac{\partial^2 u_0}{\partial y^2} - \frac{2}{5}z\frac{\partial^3 w}{\partial x^3} + \frac{3}{5}z\frac{\partial^3 w}{\partial x \partial y^2}$$
$$\eta^{(1)}_{222} = \frac{2}{5}\frac{\partial^2 v_0}{\partial y^2} - \frac{2}{5}\frac{\partial^2 u_0}{\partial x \partial y} - \frac{1}{5}\frac{\partial^2 v_0}{\partial x^2} - \frac{2}{5}z\frac{\partial^3 w}{\partial y^3} + \frac{3}{5}z\frac{\partial^3 w}{\partial x^2 \partial y}, \eta^{(1)}_{333} = \frac{1}{5}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$
$$\eta^{(1)}_{113} = \eta^{(1)}_{131} = \eta^{(1)}_{311} = -\frac{4}{15}\frac{\partial^2 w}{\partial x^2} + \frac{1}{15}\frac{\partial^2 w}{\partial y^2}, \eta^{(1)}_{223} = \eta^{(1)}_{232} = \eta^{(1)}_{322} = -\frac{4}{15}\frac{\partial^2 w}{\partial y^2} + \frac{1}{15}\frac{\partial^2 w}{\partial x^2}$$
$$\eta^{(1)}_{122} = \eta^{(1)}_{212} = \eta^{(1)}_{221} = \frac{8}{15}\frac{\partial^2 v_0}{\partial x \partial y} + \frac{4}{15}\frac{\partial^2 u_0}{\partial y^2} - \frac{1}{5}\frac{\partial^2 u_0}{\partial x^2} - \frac{4}{5}z\frac{\partial^3 w}{\partial x \partial y^2} + \frac{1}{5}z\frac{\partial^3 w}{\partial x^3}$$
$$\eta^{(1)}_{112} = \eta^{(1)}_{121} = \eta^{(1)}_{211} = \frac{8}{15}\frac{\partial^2 u_0}{\partial x \partial y} + \frac{4}{15}\frac{\partial^2 v_0}{\partial x^2} - \frac{1}{5}\frac{\partial^2 v_0}{\partial y^2} - \frac{4}{5}z\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{1}{5}z\frac{\partial^3 w}{\partial y^3}$$
$$\eta^{(1)}_{313} = \eta^{(1)}_{331} = \eta^{(1)}_{133} = -\frac{1}{5}\frac{\partial^2 u_0}{\partial x^2} - \frac{1}{15}\frac{\partial^2 u_0}{\partial y^2} - \frac{2}{15}\frac{\partial^2 v_0}{\partial x \partial y} + \frac{1}{5}z\frac{\partial^3 w}{\partial x^3} + \frac{1}{5}z\frac{\partial^3 w}{\partial x \partial y^2}$$
$$\eta^{(1)}_{332} = \eta^{(1)}_{323} = \eta^{(1)}_{233} = -\frac{1}{5}\frac{\partial^2 v_0}{\partial y^2} - \frac{1}{15}\frac{\partial^2 v_0}{\partial x^2} - \frac{2}{15}\frac{\partial^2 u_0}{\partial x \partial y} + \frac{1}{5}z\frac{\partial^3 w}{\partial y^3} + \frac{1}{5}z\frac{\partial^3 w}{\partial y \partial x^2}$$
$$\eta^{(1)}_{123} = \eta^{(1)}_{231} = \eta^{(1)}_{312} = \eta^{(1)}_{132} = \eta^{(1)}_{213} = \eta^{(1)}_{321} = -\frac{1}{3}\frac{\partial^2 w}{\partial x \partial y} \qquad \text{(A.2)}$$

The symmetric rotation gradient tensor, χ^s_{ij} $(i = 1, 2, 3)$, is

$$\chi^s_{xx} = \frac{\partial^2 w}{\partial x \partial y}, \chi^s_{yy} = -\frac{\partial^2 w}{\partial x \partial y}, \chi^s_{zz} = 0, \chi^s_{xy} = \chi^s_{yx} = \frac{1}{2}\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\right)$$
$$\chi^s_{xz} = \chi^s_{zx} = \frac{1}{4}\left(\frac{\partial^2 v_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial x \partial y}\right), \chi^s_{yz} = \chi^s_{zy} = -\frac{1}{4}\left(\frac{\partial^2 u_0}{\partial y^2} - \frac{\partial^2 v_0}{\partial x \partial y}\right), \qquad \text{(A.3)}$$

The function F is given as

$$F = a_1\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right]$$
$$+ a_2\left[\left(\frac{\partial u_0}{\partial x}\right)^2 + \left(\frac{\partial v_0}{\partial y}\right)^2\right] - 2a_3\left(\frac{\partial u_0}{\partial x}\frac{\partial^2 w}{\partial x^2} + \frac{\partial v_0}{\partial y}\frac{\partial^2 w}{\partial y^2}\right)$$
$$+ a_4\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + a_5\frac{\partial u_0}{\partial x}\frac{\partial v_0}{\partial y} - a_6\left(\frac{\partial u_0}{\partial x}\frac{\partial^2 w}{\partial y^2} + \frac{\partial v_0}{\partial y}\frac{\partial^2 w}{\partial x^2}\right)$$
$$+ a_7\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + a_8\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right)^2 - a_9\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right)\frac{\partial^2 w}{\partial x \partial y}$$
$$+ a_{10}\left[\left(\frac{\partial^2 u_0}{\partial x^2}\right)^2 + (\frac{\partial^2 v_0}{\partial y^2})^2\right] - 2a_{11}\left(\frac{\partial^2 u_0}{\partial x^2}\frac{\partial^3 w}{\partial x^3} + \frac{\partial^2 v_0}{\partial y^2}\frac{\partial^3 w}{\partial y^3}\right)$$
$$+ a_{12}\left[\left(\frac{\partial^3 w}{\partial x^3}\right)^2 + \left(\frac{\partial^3 w}{\partial y^3}\right)^2\right] + a_{13}\left[\left(\frac{\partial^3 w}{\partial x \partial y^2}\right)^2 + \left(\frac{\partial^3 w}{\partial y \partial x^2}\right)^2\right]$$
$$- a_{14}\left(\frac{\partial^2 u_0}{\partial x^2}\frac{\partial^3 w}{\partial x \partial y^2} + \frac{\partial^2 v_0}{\partial y^2}\frac{\partial^3 w}{\partial x^2 \partial y}\right) + a_{15}\left(\frac{\partial^3 w}{\partial x^3}\frac{\partial^3 w}{\partial x \partial y^2} + \frac{\partial^3 w}{\partial y^3}\frac{\partial^3 w}{\partial y \partial x^2}\right)$$

$$+a_{16}\left[\left(\frac{\partial^2 u_0}{\partial y^2}\right)^2+\left(\frac{\partial^2 v_0}{\partial x^2}\right)^2\right]+a_{17}\left(\frac{\partial^2 u_0}{\partial y^2}\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 v_0}{\partial x^2}\frac{\partial^2 v_0}{\partial y^2}\right)$$
$$-a_{19}\left(\frac{\partial^2 v_0}{\partial x\partial y}\frac{\partial^3 w}{\partial x\partial y^2}+\frac{\partial^2 u_0}{\partial x\partial y}\frac{\partial^3 w}{\partial x^2\partial y}\right)$$
$$-a_{18}\left(\frac{\partial^2 u_0}{\partial y^2}\frac{\partial^3 w}{\partial x^3}+\frac{\partial^2 v_0}{\partial x^2}\frac{\partial^3 w}{\partial y^3}-4\frac{\partial^2 u_0}{\partial y^2}\frac{\partial^3 w}{\partial x\partial y^2}-4\frac{\partial^2 v_0}{\partial x^2}\frac{\partial^3 w}{\partial x^2\partial y}\right)$$
$$+a_{20}\left[\left(\frac{\partial^2 u_0}{\partial x\partial y}\right)^2+\left(\frac{\partial^2 v_0}{\partial x\partial y}\right)^2\right]+a_{21}\left(\frac{\partial^2 u_0}{\partial x^2}\frac{\partial^2 v_0}{\partial x\partial y}+\frac{\partial^2 v_0}{\partial y^2}\frac{\partial^2 u_0}{\partial x\partial y}\right)$$
$$-a_{22}\left(\frac{\partial^3 w}{\partial x^3}\frac{\partial^2 v_0}{\partial x\partial y}+\frac{\partial^3 w}{\partial y^3}\frac{\partial^2 u_0}{\partial x\partial y}\right)+a_{23}\left(\frac{\partial^2 u_0}{\partial y^2}\frac{\partial^2 v_0}{\partial x\partial y}+\frac{\partial^2 v_0}{\partial x^2}\frac{\partial^2 u_0}{\partial x\partial y}\right) \tag{A.4}$$

in which parameters a_i ($i = 1, 2, 3 \dots, 23$) are given as

$$a_1 = c_{1(1)}A_1 + c_{2(1)}I_1 + c_{1(2)}A_2 + c_{2(2)}I_2$$
$$a_2 = c_{2(1)}A_1 + c_{2(2)}A_2, a_3 = c_{2(1)}S_1 + c_{2(2)}S_2 \tag{A.5}$$

$$a_4 = c_{3(1)}A_1 + c_{4(1)}I_1 + c_{3(2)}A_2 + c_{4(2)}I_2$$
$$a_5 = c_{4(1)}A_1 + c_{4(2)}A_2, a_6 = c_{4(1)}S_1 + c_{4(2)}S_2 \tag{A.6}$$

$$a_7 = c_{5(1)}A_1 + c_{6(1)}I_1 + c_{5(2)}A_2 + c_{6(2)}I_2$$
$$a_8 = \frac{1}{4}c_{6(1)}A_1 + \frac{1}{4}c_{6(2)}A_2, a_9 = c_{6(1)}S_1 + c_{6(2)}S_2 \tag{A.7}$$

$$a_{10} = c_{7(1)}A_1 + c_{7(2)}A_2, a_{11} = c_{7(1)}S_1 + c_{7(2)}S_2, a_{12} = c_{7(1)}I_1 + c_{7(2)}I_2 \tag{A.8}$$

$$a_{13} = c_{8(1)}I_1 + c_{8(2)}I_2, a_{14} = c_{9(1)}S_1 + c_{9(2)}S_2, a_{15} = c_{9(1)}I_1 + c_{9(2)}I_2 \tag{A.9}$$

$$a_{16} = c_{10(1)}A_1 + c_{10(2)}A_2, a_{17} = c_{11(1)}A_1 + c_{11(2)}A_2, a_{18} = c_{11(1)}S_1 + c_{11(2)}S_2 \tag{A.10}$$

$$a_{19} = c_{12(1)}S_1 + c_{12(2)}S_2, a_{20} = c_{13(1)}A_1 + c_{13(2)}A_2, a_{21} = c_{14(1)}A_1 + c_{14(2)}A_2 \tag{A.11}$$

$$a_{22} = c_{14(1)}S_1 + c_{14(2)}S_2, a_{23} = c_{15(1)}A_1 + c_{15(2)}A_2 \tag{A.12}$$

and I_i, S_i, A_i ($i = 1, 2$) are written as

$$I_1 = \int_{-h_1}^{0} z^2dz = \frac{h_1^3}{3}, S_1 = \int_{-h_1}^{0} zdz = -\frac{h_1^2}{2}, A_1 = \int_{-h_1}^{0} dz = h_1$$
$$I_2 = \int_{0}^{h_2} z^2dz = \frac{h_2^3}{3}, S_2 = \int_{0}^{h_2} zdz = \frac{h_2^2}{2}, A_2 = \int_{0}^{h_2} dz = h_2 \tag{A.13}$$

$B_{Xi}(x,y)$ and $B_{Yi}(x,y)$ ($i = 1, 2, 3, \dots, 7$) in Eq. (20) are expressed as

$$B_{X1}(x,y) = 2a_{12}\left(\frac{\partial^5 w}{\partial x^5}+3\frac{\partial^5 w}{\partial x^3\partial y^2}\right)+(a_{15}+2a_{13})\frac{\partial^5 w}{\partial x\partial y^4}-2a_1\frac{\partial^3 w}{\partial x^3}$$
$$-(a_4+2a_7)\frac{\partial^3 w}{\partial x\partial y^2}+4a_{18}\frac{\partial^4 u_0}{\partial y^4}-2a_{11}\left(2\frac{\partial^4 u_0}{\partial x^2\partial y^2}+\frac{\partial^4 u_0}{\partial x^4}\right)$$
$$-2a_{11}\frac{\partial^4 v_0}{\partial x^3\partial y}-(a_{14}+a_{19})\frac{\partial^4 v_0}{\partial x\partial y^3}+2a_3\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 v_0}{\partial x\partial y}\right)$$
$$+a_9\frac{\partial^2 u_0}{\partial y^2} \tag{A.14}$$

$$B_{Y1}(x,y) = 2a_{12}\left(\frac{\partial^5 w}{\partial y^5}+3\frac{\partial^5 w}{\partial x^2\partial y^3}\right)+(a_{15}+2a_{13})\frac{\partial^5 w}{\partial x^4\partial y}-2a_1\frac{\partial^3 w}{\partial y^3}$$
$$-(a_4+2a_7)\frac{\partial^3 w}{\partial x^2\partial y}+4a_{18}\frac{\partial^4 v_0}{\partial x^4}-2a_{11}(2\frac{\partial^4 v_0}{\partial x^2\partial y^2}+\frac{\partial^4 v_0}{\partial y^4})$$
$$-2a_{11}\frac{\partial^4 u_0}{\partial x\partial y^3}-(a_{14}+a_{19})\frac{\partial^4 u_0}{\partial x^3\partial y}+2a_3\left(\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial^2 u_0}{\partial x\partial y}\right)$$
$$+a_9\frac{\partial^2 v_0}{\partial x^2} \tag{A.15}$$

$$B_{X2}(x,y) = -2a_{12}\frac{\partial^4 w}{\partial x^4}-(2a_{13}+a_{15})\frac{\partial^4 w}{\partial x^2\partial y^2}-a_{15}\frac{\partial^4 w}{\partial y^4}+a_4\frac{\partial^2 w}{\partial y^2}$$
$$+2a_1\frac{\partial^2 w}{\partial x^2}+a_{14}\frac{\partial^3 v_0}{\partial y^3}+2a_{11}\frac{\partial^3 v_0}{\partial x^2\partial y}+2a_{11}\frac{\partial^3 u_0}{\partial x^3}$$
$$+(a_{18}+a_{19})\frac{\partial^3 u_0}{\partial x\partial y^2}-2a_3\frac{\partial u_0}{\partial x}-a_6\frac{\partial v_0}{\partial y} \tag{A.16}$$

$$B_{Y2}(x,y) = -2a_{12}\frac{\partial^4 w}{\partial y^4}-(2a_{13}+a_{15})\frac{\partial^4 w}{\partial x^2\partial y^2}-a_{15}\frac{\partial^4 w}{\partial x^4}+a_4\frac{\partial^2 w}{\partial x^2}$$
$$+2a_1\frac{\partial^2 w}{\partial y^2}+a_{14}\frac{\partial^3 u_0}{\partial x^3}+2a_{11}\frac{\partial^3 u_0}{\partial x\partial y^2}+2a_{11}\frac{\partial^3 v_0}{\partial y^3}$$
$$+(a_{18}+a_{19})\frac{\partial^3 v_0}{\partial x^2\partial y}-2a_3\frac{\partial v_0}{\partial y}-a_6\frac{\partial u_0}{\partial x} \tag{A.17}$$

$$B_{X3}(x,y) = 2a_{12}\frac{\partial^3 w}{\partial x^3}+a_{15}\frac{\partial^3 w}{\partial x\partial y^2}-2a_{11}\frac{\partial^2 u_0}{\partial x^2}-a_{18}\frac{\partial^2 u_0}{\partial y^2}-a_{22}\frac{\partial^2 v_0}{\partial x\partial y} \tag{A.18}$$

$$B_{Y3}(x,y) = 2a_{12}\frac{\partial^3 w}{\partial y^3}+a_{15}\frac{\partial^3 w}{\partial y\partial x^2}-2a_{11}\frac{\partial^2 v_0}{\partial y^2}-a_{18}\frac{\partial^2 v_0}{\partial x^2}-a_{22}\frac{\partial^2 u_0}{\partial x\partial y} \tag{A.19}$$

$$B_{X4}(x,y) = 2a_{11}\frac{\partial^4 w}{\partial x^4}+(a_{14}+a_{19})\frac{\partial^4 w}{\partial x^2\partial y^2}+a_{22}\frac{\partial^4 w}{\partial y^4}-2a_3\frac{\partial^2 w}{\partial x^2}$$
$$-a_6\frac{\partial^2 w}{\partial y^2}-(a_{17}+2a_{20})\frac{\partial^3 u_0}{\partial x\partial y^2}-2a_{10}\frac{\partial^3 u_0}{\partial x^3}$$
$$-(a_{21}+a_{23})\frac{\partial^3 v_0}{\partial x^2\partial y}-a_{21}\frac{\partial^3 v_0}{\partial y^3}+2a_2\frac{\partial u_0}{\partial x}+a_5\frac{\partial v_0}{\partial y} \tag{A.20}$$

$$B_{Y4}(x,y) = 2a_{11}\frac{\partial^4 w}{\partial y^4}+(a_{14}+a_{19})\frac{\partial^4 w}{\partial x^2\partial y^2}+a_{22}\frac{\partial^4 w}{\partial x^4}-2a_3\frac{\partial^2 w}{\partial y^2}$$
$$-a_6\frac{\partial^2 w}{\partial x^2}-(a_{17}+2a_{20})\frac{\partial^3 v_0}{\partial x^2\partial y}-2a_{10}\frac{\partial^3 v_0}{\partial y^3}$$
$$-(a_{21}+a_{23})\frac{\partial^3 u_0}{\partial x\partial y^2}-a_{21}\frac{\partial^3 u_0}{\partial x^3}+2a_2\frac{\partial v_0}{\partial y}+a_5\frac{\partial u_0}{\partial x} \tag{A.21}$$

$$B_{X5}(x,y) = (a_{18}+a_{19})\frac{\partial^4 w}{\partial x\partial y^3}+2a_{11}\frac{\partial^4 w}{\partial x^3\partial y}-a_9\frac{\partial^2 w}{\partial x\partial y}$$
$$-(a_{17}+2a_{20})\frac{\partial^3 v_0}{\partial x\partial y^2}-2a_{16}\frac{\partial^3 v_0}{\partial x^3}-(a_{21}+a_{23})\frac{\partial^3 u_0}{\partial x^2\partial y}$$
$$-a_{23}\frac{\partial^3 u_0}{\partial y^3}+2a_8\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right) \tag{A.22}$$

$$B_{Y5}(x,y) = (a_{18}+a_{19})\frac{\partial^4 w}{\partial x^3\partial y}+2a_{11}\frac{\partial^4 w}{\partial x\partial y^3}-a_9\frac{\partial^2 w}{\partial x\partial y}$$
$$-(a_{17}+2a_{20})\frac{\partial^3 u_0}{\partial x^2\partial y}-2a_{16}\frac{\partial^3 u_0}{\partial y^3}-(a_{21}+a_{23})\frac{\partial^3 v_0}{\partial x\partial y^2}$$
$$-a_{23}\frac{\partial^3 v_0}{\partial x^3}+2a_8\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right) \tag{A.23}$$

$$B_{X6}(x,y) = -a_{18}\left(\frac{\partial^3 w}{\partial y^3} - 4\frac{\partial^3 w}{\partial x^2 \partial y}\right) + 2a_{16}\frac{\partial^2 v_0}{\partial x^2} + a_{17}\frac{\partial^2 v_0}{\partial y^2} + a_{23}\frac{\partial^2 u_0}{\partial x \partial y} \tag{A.24}$$

$$B_{Y6}(x,y) = -a_{18}\left(\frac{\partial^3 w}{\partial x^3} - 4\frac{\partial^3 w}{\partial x \partial y^2}\right) + 2a_{16}\frac{\partial^2 u_0}{\partial y^2} + a_{17}\frac{\partial^2 u_0}{\partial x^2} + a_{23}\frac{\partial^2 v_0}{\partial x \partial y} \tag{A.25}$$

$$B_{X7}(x,y) = -2a_{11}\frac{\partial^3 w}{\partial x^3} - a_{14}\frac{\partial^3 w}{\partial x \partial y^2} + 2a_{10}\frac{\partial^2 u_0}{\partial x^2} + a_{17}\frac{\partial^2 u_0}{\partial y^2} + a_{21}\frac{\partial^2 v_0}{\partial x \partial y} \tag{A.26}$$

$$B_{Y7}(x,y) = -2a_{11}\frac{\partial^3 w}{\partial y^3} - a_{14}\frac{\partial^3 w}{\partial x^2 \partial y} + 2a_{10}\frac{\partial^2 v_0}{\partial y^2} + a_{17}\frac{\partial^2 v_0}{\partial x^2} + a_{21}\frac{\partial^2 u_0}{\partial x \partial y} \tag{A.27}$$

$B_{xi}(x,y)$ and $B_{yi}(x,y)$ ($i = 2, 3, 4, 6, 7$), which are simplified from $B_{Xi}(x,y)$ and $B_{Yi}(x,y)$ ($i = 2, 3, 4, 6, 7$), are expressed as

$$B_{x2}(x,y) = -2a_{12}\frac{\partial^4 w}{\partial x^4} + 2a_1\frac{\partial^2 w}{\partial x^2} + 2a_{11}\frac{\partial^3 u_0}{\partial x^3} + (a_{18} + a_{19})\frac{\partial^3 u_0}{\partial x \partial y^2} - 2a_3\frac{\partial u_0}{\partial x} \tag{A.28}$$

$$B_{y2}(x,y) = -2a_{12}\frac{\partial^4 w}{\partial y^4} + 2a_1\frac{\partial^2 w}{\partial y^2} + 2a_{11}\frac{\partial^3 v_0}{\partial y^3} + (a_{18} + a_{19})\frac{\partial^3 v_0}{\partial x^2 \partial y} - 2a_3\frac{\partial v_0}{\partial y} \tag{A.29}$$

$$B_{x3}(x,y) = 2a_{12}\frac{\partial^3 w}{\partial x^3} - 2a_{11}\frac{\partial^2 u_0}{\partial x^2} - a_{18}\frac{\partial^2 u_0}{\partial y^2} \tag{A.30}$$

$$B_{y3}(x,y) = 2a_{12}\frac{\partial^3 w}{\partial y^3} - 2a_{11}\frac{\partial^2 v_0}{\partial y^2} - a_{18}\frac{\partial^2 v_0}{\partial x^2} \tag{A.31}$$

$$B_{x4}(x,y) = 2a_{11}\frac{\partial^4 w}{\partial x^4} - 2a_3\frac{\partial^2 w}{\partial x^2} - (a_{17} + 2a_{20})\frac{\partial^3 u_0}{\partial x \partial y^2} - 2a_{10}\frac{\partial^3 u_0}{\partial x^3} + 2a_2\frac{\partial u_0}{\partial x} \tag{A.32}$$

$$B_{y4}(x,y) = 2a_{11}\frac{\partial^4 w}{\partial y^4} - 2a_3\frac{\partial^2 w}{\partial y^2} - (a_{17} + 2a_{20})\frac{\partial^3 v_0}{\partial x^2 \partial y} - 2a_{10}\frac{\partial^3 v_0}{\partial y^3} + 2a_2\frac{\partial v_0}{\partial y} \tag{A.33}$$

$$B_{x6}(x,y) = 2a_{16}\frac{\partial^2 v_0}{\partial x^2} + a_{23}\frac{\partial^2 u_0}{\partial x \partial y} \tag{A.34}$$

$$B_{y6}(x,y) = 2a_{16}\frac{\partial^2 u_0}{\partial y^2} + a_{23}\frac{\partial^2 v_0}{\partial x \partial y} \tag{A.35}$$

$$B_{x7}(x,y) = -2a_{11}\frac{\partial^3 w}{\partial x^3} + 2a_{10}\frac{\partial^2 u_0}{\partial x^2} + a_{17}\frac{\partial^2 u_0}{\partial y^2} \tag{A.36}$$

$$B_{y7}(x,y) = -2a_{11}\frac{\partial^3 w}{\partial y^3} + 2a_{10}\frac{\partial^2 v_0}{\partial y^2} + a_{17}\frac{\partial^2 v_0}{\partial x^2} \tag{A.37}$$

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