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A reformulation of constitutive relations in the strain gradient elasticity theory for isotropic materials

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ABSTRACT

The general isotropic strain gradient elasticity theory with five higher-order elastic constants is reformulated by introducing two different orthogonal decompositions of the strain gradient tensor. Just applying the mathematical reformulations, no extra conditions needed, the constitutive relations, equilibrium equation and boundary conditions are reformulated. In the reformulated theory, the number of independent higher-order elastic constants is proved to be three for isotropic materials, which indicates that the five higher-order elastic constants in the general isotropic strain gradient elasticity theory are dependent with each other. Therefore, the general strain gradient elasticity theory contains only three independent material length-scale parameters for isotropic materials in addition to the Lame constants. The new theory is different from the existed strain gradient elasticity theory with one or three material length-scale parameters, which introduces extra conditions during deriving process. Moreover, the reformulated theory can be directly reduced to that of incompressible materials by assuming the terms associated with hydrostatic strains to be zero. Some examples, such as torsion of cylindrical bars, shearing of fixed-end layers, and pure bending of thin beams, are performed to reveal the necessity of including multi-length-scale parameters in the strain gradient elasticity theory to predict size effects at micron scale.

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1. Introduction

Many experiments have shown the size-dependent deformation behaviors in micron scale. In the non-uniform plastic deformation, the size effects have been observed in the experiments of measuring micro-indentation hardness of metallic materials (Ma and Clarke, 1995), shear strength of copper wires in torsion (Fleck et al., 1994), and bend moments of ultra-thin beams (Stolken and Evans, 1998). By contrast, in elastic deformation, the size dependence of the normalized bending rigidity exists in micro-beams of both metals and polymers (Guo et al., 2005; Tang and Alici, 2011a, 2011b; Lam et al., 2003). Because of the lack of internal length-scale parameters, the classical elasticity and plasticity theories fail to describe such a behavior in the micron scale. Meanwhile, load or geometrically induced stress singularities cannot be properly accounted by the standard continuum mechanical models. However, this is possible with the use of higher-order continuum mechanics theories, where intrinsic parameters correlating the microstructure and the macrostructure are involved in the constitutive relations. Different versions of higher-order continuum mechanics have been developed by many authors. Relationships between those theories have been discussed by Tekoglu and Onck (2008).

Generally, higher-order theories can be classified into couple stress theories and general strain gradient theories, according to the deformation metrics used. In the classical couple stress theory (Toupin, 1962; Mindlin and Tiersten, 1962; Koiter, 1964), only the gradient of the rotation vector enters the strain energy density function, and, hence, two additional material parameters are introduced besides the Lame constants for isotropic materials. By introducing a so-called equilibrium condition of moments of couples (the couple of force couples) to force the couple stress tensor to be symmetric, Yang et al. (2002) modified the classical couple stress theory to include only one additional material parameter. Recently, however, Hadjesfandiari and Dargush (2011) came to a contrary conclusion that the couple stress tensor is of the skew-symmetric character.

The more general strain gradient elastic theory including all components of the higher-order deformation is proposed by Mindlin (1964) to describe the linear elastic behavior of microstructures. This theory requires 16 additional length constants for isotropic materials in addition to two Lame constants. The application of this theory is limited as it requires the formidable task of determining,

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theoretically or experimentally, 16 additional constants. For practical purposes, Mindlin and Eshel (1968) further formulated three simpler versions of the general isotropic theory, utilizing only two material and five internal length-scale constants in the final constitutive relation rather than 18 used in Mindlin's initial model. One of these versions uses the classical strains and the second-order gradient of displacement as the deformation metrics. In the second version, the second-order gradient of displacement is replaced by the gradient of strain, and in the third version, the deformation variables include the classical strains, the gradient of rotation, and the fully symmetric part of the gradient of strain. Although the simpler versions reduce independent length-scale parameters from 16 to five for isotropic materials, the application of this theory in engineering is limited as five length-scale parameters are still very difficult to be determined experimentally. Fleck and Hutchinson (1993, 2001) extended the first version of Mindlin theory to plasticity and proposed a deformation theory of strain gradient plasticity, which involves three length-scale parameters. Lam et al. (2003) reformulated the theory by applying a set of higher-order metrics to characterize strain gradient behaviors, and proposed a isotropic strain gradient elasticity theory. In their theory, the equilibrium condition of moments of couples is applied to force the symmetric character of the couple stress tensor and reduce the number of elastic length-scale parameters from five to three. However, in our opinion, the mechanical effect of the moment couple is unable to be comprehended for the free character of moment vectors.

In addition to the simplified model of the general strain gradient elasticity theory developed by Mindlin, a simple model of isotropic strain gradient elasticity with only one length-scale parameter has been formulated by Aifantis (1992), in which classical stresses are related to classical strains and the Laplacian gradient of strains. Subsequently, this theory has been extended with additional terms to account for surface effects (Vardoulakis and Sulem, 1995; Vardoulakis et al., 1996; Exadaktylos, 1998). The theory proposed by Aifantis can be formally obtained as a special case of the Mindlin theory (Li et al, 2004; Lazar and Maugin, 2005; Askes and Aifantis, 2011). Although one length-scale parameter can be expediently determined by simple bending or torsion test, it has been demonstrated that the strain gradient plasticity theory with a single length parameter does not have a scope to include the wide range of small-scale phenomena (Fleck and Hutchinson, 2001). Therefore, the strain gradient theory with multiple length parameters is necessary to capture the size effects of mechanic behaviors at the micron scale.

Recently, strain gradient theory has new development. Polizzotto (2012) proposed a gradient elasticity theory for continua featured by not only a strain energy depending on the strain and the first-order strain gradient, but also a kinetic energy depending on the velocity and the first-order velocity gradient, in which the effects of both strain gradient and higher-order inertia are combined. Further, this theory has been even extended to the second-order strain gradient elasticity with second-order velocity gradient inertia (Polizzotto, 2013). Moreover, Auffray et al. (2013) derived and provided the explicit matrix representations of the sixth-order elastic tensor for all the three-dimensional (3D) anisotropic cases in a compact and well-structured manner. In addition, Mühlich et al. (2012) developed an alternative method for the approximation of the material properties in linear elastic strain gradient effective media. Bacca et al. (2013) provided an analytical approach to the determination of the parameters defining an elastic higher-order (Mindlin) material as the homogenization of a heterogeneous Cauchy elastic material. Although many new achievements have been made, the strain gradient theory should be contributed more as the basis to form a unified and effective theory for application.

The purpose of this article is to propose a general strain gradient elasticity theory by reformulating the constitutive relations in terms of two sets of independent higher-order metrics and determine the number of independent material constants needed for an isotropic material in the general strain gradient elasticity theory. The rest is organized as follows. Section 2 reviews the general isotropic strain gradient theory. In Section 3, two new sets of independent higher-order deformation metrics are developed to split the strain gradient tensor into mutually independent parts and the corresponding work-conjugated higher-order stress tensors are defined. In Section 4, the constitutive relations are reformulated and the number of independent higher-order material constants is proved to be three for isotropic linear elastic materials. Then, the general isotropic strain gradient elasticity theory containing three higher-order elastic constants is re-expressed in the form of strain gradient components, and the equilibrium relations and boundary conditions are derived by applying the variational principle of the strain gradient theory. Section 5 presents the contribution of each strain gradient component and the influence of higher-order length-scale parameters through three basic problems. Finally, conclusions are summarized in Section 6.

2. Review of general strain gradient elasticity theory

In the general strain gradient elasticity theory (Mindlin and Eshel, 1968), the total strain energy density is a function of strain and its first-order gradient, given by

$$w = w(\varepsilon_{ij}, \eta_{ijk}), \tag{1}$$

where ε_{ij} is the symmetric strain tensor and η_{ijk} is the strain gradient tensor with the minor symmetry in the last two indices. The strain tensor and strain gradient tensor are defined, respectively, as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2}$$

$$\eta_{ijk} = \varepsilon_{kj,i}, \tag{3}$$

where u_i is the displacement vector and a comma denotes the differentiation with respect to the coordinates. Then, the corresponding stress σ_{ij} and the higher-order stress $\tau_{ijk}(=\tau_{ikj})$ work-conjugated to the strain ε_{ij} and the strain gradient η_{ijk}, respectively, can be written as

$$\sigma_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}}, \quad \tau_{ijk} = \frac{\partial w}{\partial \eta_{ijk}}. \tag{4}$$

For a volume V of a solid with boundary S and sharp edge C, the principle of virtual work for the strain gradient theory is

$$\int_V (\sigma_{ij}\delta\varepsilon_{ij} + \tau_{ijk}\delta\eta_{ijk})\mathrm{d}V = \int_V \bar{b}_k \delta u_k \mathrm{d}V + \int_S \left(\bar{t}_k \delta u_k + \bar{r}_k D\delta u_k\right)\mathrm{d}S + \oint_C \bar{f}_k \delta u_k \mathrm{d}C, \tag{5}$$

where $\bar{b}_k$ is the body force per unit volume, $\bar{t}_k$ is the surface traction, $\bar{r}_k$ is the surface double-force traction, $\bar{f}_k$ is the line load along the sharp edge, and $D = n_i\partial_i$ denotes the normal gradient operator. The equilibrium equation in the body V can be derived using the virtual work principle as

$$\sigma_{ik,i} - \tau_{ijk,ij} + \bar{b}_k = 0, \tag{6}$$

and the boundary conditions on S and along C are, respectively,

$$\bar{t}_k = n_i\left(\sigma_{ik} - \tau_{ijk,j}\right) + (D_p n_p) n_i n_j \tau_{ijk} - D_i\left(n_j \tau_{ijk}\right) \text{ or } \bar{u}_k = u_k, \tag{7}$$

$$\bar{r}_k = n_i n_j \tau_{ijk} \text{ or } \overline{Du_k} = Du_k, \tag{8}$$

and

$$\bar{f}_k = [n_i k_j \tau_{ijk}] \text{ or } \bar{u}_k = u_k, \tag{9}$$

where n_i is a unit vector normal to the boundary surface S, $D_i = (\delta_{ik} - n_i n_k)\partial_k$ is the surface gradient operator, and k_j is the outer co-normal vector satisfying the following relation

$$k_j = e_{ikj} s_i n_k, \tag{10}$$

with the alternating tensor e_{ikj} and the unit vector s_i tangent to the edge C. The square brackets in Eq. (9) represent the difference between the values of the enclosed quantity on the two sides of the edge.

3. Development of independent higher-order metrics

In this section, two sets of independent higher-order deformation metrics will be developed by introducing two different orthogonal decompositions of the strain gradient tensor, of which one is based on a hydrostatic/deviatoric splitting, the other on a symmetric/anti-symmetric splitting.

For the hydrostatic/deviatoric splitting, it is known that the strain tensor can be expressed as its spherical and deviatoric parts

$$\varepsilon_{ij} = \frac{1}{3}\delta_{ij}\varepsilon_{nn} + \varepsilon'_{ij}, \tag{11}$$

where δ_{ij} is the Kronecker delta, $\frac{1}{3}\varepsilon_{nn}$ is the mean or hydrostatic strain associated with a change in volume, and ε'_{ij} is the deviatoric strain tensor associated with a change in shape. Following Eq. (11), the strain gradient tensor η_{ijk} can also be decomposed into a hydrostatic part η^h_{ijk} and a deviatoric part η'_{ijk} as

$$\eta_{ijk} = \eta^h_{ijk} + \eta'_{ijk}, \tag{12}$$

where

$$\eta^h_{ijk} = \frac{1}{3}\delta_{jk}\varepsilon_{nn,i} = \frac{1}{3}\delta_{jk}\eta_{inn}, \ \eta'_{ijk} = \eta_{ijk} - \eta^h_{ijk} = \varepsilon'_{kj,i}. \tag{13}$$

Here, the deviatoric part η'_{ijk} is the deviatoric strain gradient $\varepsilon'_{kj,i}$ associated with the shape change in strain gradient, and, in fact, equals the plastic strain gradient introduced by Fleck and Hutchinson (2001). In accordance with the decomposition of plastic strain gradients into three parts proposed by Fleck and Hutchinson (2001), the strain gradient η_{ijk} can be decomposed into four independent components as

$$\eta_{ijk} = \eta^h_{ijk} + \eta'^{(1)}_{ijk} + \eta'^{as}_{ijk} + \eta'^{(2)}_{ijk}, \tag{14}$$

with

$$\eta'^{(1)}_{ijk} = \eta'^{s}_{ijk} - \frac{2}{15}\left(\delta_{ij}e_{kpq} + \delta_{jk}e_{ipq} + \delta_{ki}e_{jpq}\right)\chi'^{a}_{pq}, \tag{15}$$

$$\eta'^{as}_{ijk} = \frac{1}{3}e_{ijp}\chi'^{s}_{pk} + \frac{1}{3}e_{ikp}\chi'^{s}_{pj}, \tag{16}$$

$$\eta'^{(2)}_{ijk} = \frac{1}{3}e_{ijp}\chi'^{a}_{pk} + \frac{1}{3}e_{ikp}\chi'^{a}_{pj} + \frac{2}{15}\left(\delta_{ij}e_{kpq} + \delta_{jk}e_{ipq} + \delta_{ki}e_{jpq}\right)\chi'^{a}_{pq}, \tag{17}$$

where

$$\eta'^{s}_{ijk} = \frac{1}{3}\left(\eta'_{ijk} + \eta'_{kij} + \eta'_{jki}\right), \ \chi'_{ij} = e_{ipq}\eta'_{pqj} = e_{ipq}\varepsilon'_{jq,p}$$
$$\chi'^{s}_{ij} = \frac{1}{2}\left(\chi'_{ij} + \chi'_{ji}\right), \ \chi'^{a}_{ij} = \frac{1}{2}\left(\chi'_{ij} - \chi'_{ji}\right). \tag{18}$$

Accordingly, the decomposition of the higher-order stress tensor τ_{ijk}, work-conjugate to the strain gradient tensor η_{ijk}, can be written as

$$\tau_{ijk} = \tau^h_{ijk} + \tau'^{(1)}_{ijk} + \tau'^{as}_{ijk} + \tau'^{(2)}_{ijk}, \tag{19}$$

where the components τ^h_{ijk}, $\tau'^{(1)}_{ijk}$, τ'^{as}_{ijk}, and $\tau'^{(2)}_{ijk}$, work-conjugate to the strain gradient components, η^h_{ijk}, $\eta'^{(1)}_{ijk}$, η'^{as}_{ijk}, and $\eta'^{(2)}_{ijk}$, respectively, are defined as

$$\tau^h_{ijk} = \frac{1}{3}\delta_{jk}\tau_{inn}, \tag{20}$$

$$\tau'^{(1)}_{ijk} = \tau'^{s}_{ijk} - \frac{1}{5}\left(\delta_{ij}\tau'^{s}_{mmk} + \delta_{jk}\tau'^{s}_{mmi} + \delta_{ki}\tau'^{s}_{mmj}\right), \tag{21}$$

$$\tau'^{as}_{ijk} = \frac{1}{6}\left(2\tau'_{ijk} - \tau'_{jik} - \tau'_{kji} + e_{ijp}e_{kts}\tau'_{tsp} + e_{ikp}e_{jts}\tau'_{tsp}\right), \tag{22}$$

$$\tau'^{(2)}_{ijk} = \frac{1}{5}\left(\delta_{ij}\tau'^{s}_{mmk} + \delta_{jk}\tau'^{s}_{mmi} + \delta_{ki}\tau'^{s}_{mmj}\right) + \frac{1}{6}\left(2\tau'_{ijk} - \tau'_{jik} - \tau'_{kji} - e_{ijp}e_{kts}\tau'_{tsp} - e_{ikp}e_{jts}\tau'_{tsp}\right), \tag{23}$$

with

$$\tau'^{s}_{ijk} = \frac{1}{3}\left(\tau'_{ijk} + \tau'_{jki} + \tau'_{kij}\right), \quad \tau'_{ijk} = \tau_{ijk} - \tau^h_{ijk}, \tag{24}$$

where τ'^{s}_{ijk} is the symmetric component of the deviatoric higher-order stress tensor, and the deviatoric part τ'_{ijk} of higher-order stress is work-conjugate to the deviatoric strain gradient η'_{ijk}.

For the symmetric/anti-symmetric splitting, the strain gradient tensor η_{ijk} can be directly decomposed into its symmetric and anti-symmetric parts, η^s_{ijk} and η^a_{ijk}, (Fleck and Hutchinson, 1997), respectively, as follows:

$$\eta^s_{ijk} = \frac{1}{3}\left(\eta_{ijk} + \eta_{jki} + \eta_{kij}\right), \ \eta^a_{ijk} = \eta_{ijk} - \eta^s_{ijk} = \frac{1}{3}e_{ijp}\chi_{pk} + \frac{1}{3}e_{ikp}\chi_{pj}, \tag{25}$$

where $\chi_{ij} = e_{ipq}\eta_{pqj}$ is the curvature tensor. Then, the symmetric part η^s_{ijk} can be further split into a trace part, $\eta^{(0)}_{ijk}$, and a traceless part, $\eta^{(1)}_{ijk}$, (Lam et al., 2003) and, moreover, new independent strain gradient metrics can be obtained by splitting the anti-symmetric part η^a_{ijk} into two independent parts, η^{as}_{ijk} and η^{aa}_{ijk}, according to the decomposition of the curvature tensor χ_{ij} into its symmetric part χ^s_{ij} and anti-symmetric part χ^a_{ij}. Thus, the new set of independent components is given by

$$\eta_{ijk} = \eta^{(0)}_{ijk} + \eta^{(1)}_{ijk} + \eta^{as}_{ijk} + \eta^{aa}_{ijk}, \tag{26}$$

where

$$\eta^{(0)}_{ijk} = \frac{1}{5}\left(\delta_{ij}\eta^s_{mmk} + \delta_{jk}\eta^s_{mmi} + \delta_{ki}\eta^s_{mmj}\right), \tag{27}$$

$$\eta^{(1)}_{ijk} = \eta^s_{ijk} - \eta^{(0)}_{ijk}, \tag{28}$$

$$\eta^{as}_{ijk} = \frac{1}{3}e_{ijp}\chi^s_{pk} + \frac{1}{3}e_{ikp}\chi^s_{pj}, \tag{29}$$

and

$$\eta^{aa}_{ijk} = \frac{1}{3}e_{ijp}\chi^a_{pk} + \frac{1}{3}e_{ikp}\chi^a_{pj}, \tag{30}$$

with

$$\chi^s_{ij} = \frac{1}{2}\left(\chi_{ij} + \chi_{ji}\right), \ \chi^a_{ij} = \frac{1}{2}\left(\chi_{ij} - \chi_{ji}\right). \tag{31}$$

By defining the corresponding higher-order stress components $\tau^{(0)}_{ijk}$, $\tau^{(1)}_{ijk}$, τ^{as}_{ijk}, and τ^{aa}_{ijk}, work-conjugate to the set of higher-order deformation metrics $\eta^{(0)}_{ijk}$, $\eta^{(1)}_{ijk}$, η^{aa}_{ijk}, and η^{as}_{ijk}, respectively, the higher-order stress tensor can also be expressed as

$$\tau_{ijk} = \tau^{(0)}_{ijk} + \tau^{(1)}_{ijk} + \tau^{as}_{ijk} + \tau^{aa}_{ijk}, \tag{32}$$

where $\tau^{(0)}_{ijk}$ and $\tau^{(1)}_{ijk}$ are the trace and traceless parts, respectively, split from the symmetric part of the higher-order stress tensor $\tau^s_{ijk} = \frac{1}{3}(\tau_{ijk} + \tau_{jki} + \tau_{kij})$, which is work-conjugate to the symmetric part of strain gradient tensor η^s_{ijk}. While τ^{as}_{ijk} and τ^{aa}_{ijk} are other two independent parts split from the anti-symmetric part of the higher-order stress tensor $\tau^a_{ijk} = \tau_{ijk} - \tau^s_{ijk}$, which is work-conjugate to the anti-symmetric part of strain gradient tensor η^a_{ijk}.

In the above-presented discussions, the two sets of higher-order deformation metrics have been given in the form of different orthogonal components in Eqs. (14) and (26). The difference between them is that the latter three terms in the first set are independent of the dilatational deformation, but the first term depends on dilatational deformation; nevertheless, the second and third terms in the second set are independent of the dilatational deformation, but the first and last terms are related to the dilatational deformation. With the help of the relations $\varepsilon_{kj,i} = \varepsilon'_{kj,i} + \frac{1}{3}\delta_{jk}\varepsilon_{nn,i}$, $\varepsilon_{ni,n} = \varepsilon'_{ni,n} + \frac{1}{3}\varepsilon_{nn,i}$, and $\chi^s_{ij} = \chi'^s_{ij}$, it is found that the components $\eta^{(1)}_{ijk}$ and η^{as}_{ijk} of the strain gradient tensor are equal to the components $\eta'^{(1)}_{ijk}$ and η'^{as}_{ijk} of the deviatoric strain gradient tensor, respectively, that is $\eta'^{(1)}_{ijk} = \eta^{(1)}_{ijk}$ and $\eta'^{as}_{ijk} = \eta^{as}_{ijk}$. Correspondingly, the components $\tau^{(1)}_{ijk}$ and τ^{as}_{ijk} of the higher-order stress tensor are equal to the components $\tau'^{(1)}_{ijk}$ and τ'^{as}_{ijk} of the deviatoric higher-order stress tensor, respectively, that is $\tau'^{(1)}_{ijk} = \tau^{(1)}_{ijk}$ and $\tau'^{as}_{ijk} = \tau^{as}_{ijk}$. And, hence, no differentiation will be made between $\eta'^{(1)}_{ijk}$ and $\eta^{(1)}_{ijk}$, η'^{as}_{ijk} and η^{as}_{ijk}, $\tau'^{(1)}_{ijk}$ and $\tau^{(1)}_{ijk}$, τ'^{as}_{ijk} and τ^{as}_{ijk} in the rest of the current paper. Moreover, it is confirmed that the traceless part, $\eta^{(1)}_{ijk}$ of the strain gradient tensor is the same as provided by Lam et al. (2003), who refer to it as the deviatoric stretch gradient. In addition, it is also shown that all independent components of both strain gradient tensor and the higher-order stress tensor carry the minor symmetry of the strain gradient tensor.

4. Reformulation of constitutive relations

4.1. Independent higher-order material constants

For linear elastic isotropic materials, the total strain energy density in the strain gradient theory (Eq. (1)) consists of the conventional part w_c, depending on strains, and the higher-order part w_h, depending on strain gradients. Thus, the total strain energy density can be expressed as

$$w = w_c + w_h = \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \frac{1}{2}F_{ijklpq}\eta_{ijk}\eta_{lpq}, \tag{33}$$

where $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})$ is the conventional elastic tensor with the Lame constants λ and μ, and F_{ijklpq} is the sixth-order isotropic elastic tensor obeying the symmetry $F_{ijklpq} = F_{lpqijk}$ and, moreover, requires the symmetries $F_{ijklpq} = F_{ikjlpq} = F_{ijklqp}$ because of the minor symmetry of the strain gradient tensor η_{ijk}. The resulting constitutive relations can be derived using Eq. (4), as follows:

$$\begin{aligned}\sigma_{ij} &= C_{ijkl}\varepsilon_{kl} = \lambda\delta_{ij}\varepsilon_{nn} + 2\mu\varepsilon_{ij} \\ \tau_{ijk} &= F_{ijklpq}\eta_{lpq}.\end{aligned} \tag{34}$$

Because of the development of the new independent higher-order metrics, the higher-order part w_h in the total strain energy density can be expressed in terms of the differential independent components of the strain gradient tensor. On the one hand, for the hydrostatic and deviatoric parts of the strain gradient tensor in Eq. (12), the higher-order part has the following form:

$$w_h = \frac{1}{2}F^h_{ijklpq}\eta^h_{lpq}\eta^h_{ijk} + \frac{1}{2}F^d_{ijklpq}\eta'_{lpq}\eta'_{ijk}, \tag{35}$$

where the sixth-order isotropic tensors F^h_{ijklpq} and F^d_{ijklpq} have a symmetry similar to that of F_{ijklpq}; however, in addition, the minor symmetric property of the hydrostatic strain gradient tensor η^h_{ijk} enforces that the symmetry of the tensor F^h_{ijklpq} with respect to its indices (j, k) depends on δ_{jk}, and the property of the deviatoric tensor η'_{ijk} requires the tensor F^d_{ijklpq} has $F^d_{isslpq} = 0$.

On the other hand, according to the symmetric and anti-symmetric parts of the strain gradient tensor in Eq. (25), the higher-order part of strain energy density can also be written as

$$w_h = \frac{1}{2}F^s_{ijklpq}\eta^s_{lpq}\eta^s_{ijk} + \frac{1}{2}F^a_{ijklpq}\eta^a_{lpq}\eta^a_{ijk}, \tag{36}$$

in which the sixth-order isotropic tensors F^s_{ijklpq} and F^a_{ijklpq} have the major symmetry, $F^s_{ijklpq} = F^s_{kijlpq} = F^s_{jkilpq}$, and the major anti-symmetry, $F^a_{ijklpq} + F^a_{kijlpq} + F^a_{jkilpq} = 0$, in addition to obeying the symmetry of F_{ijklpq}.

The sixth-order isotropic tensors having their own properties on the symmetries or anti-symmetries can be read as a linear combination of their basic tensors provided in Appendix. From Eqs. (A.5)–(A.9), the sixth-order isotropic tensors, F_{ijklpq}, F^h_{ijklpq}, F^d_{ijklpq}, F^s_{ijklpq}, and F^a_{ijklpq} can be expressed, respectively, using their own basic tensors, as

$$\begin{aligned}F_{ijklpq} = a_1(\mathbf{S}_1)_{ijklpq} &+ a_2(\mathbf{S}_2)_{ijklpq} + a_3(\mathbf{S}_3)_{ijklpq} + a_4(\mathbf{S}_4)_{ijklpq} \\ &+ a_5(\mathbf{S}_5)_{ijklpq},\end{aligned} \tag{37}$$

$$F^h_{ijklpq} = d(\mathbf{S}_3)_{ijklpq}, \tag{38}$$

$$F^d_{ijklpq} = d_1(\mathbf{D}_1)_{ijklpq} + d_2(\mathbf{D}_2)_{ijklpq} + d_3(\mathbf{D}_3)_{ijklpq}, \tag{39}$$

$$F^s_{ijklpq} = b_1(\mathbf{K}_1)_{ijklpq} + b_2(\mathbf{K}_2)_{ijklpq}, \tag{40}$$

$$F^a_{ijklpq} = c_1(\mathbf{A}_1)_{ijklpq} + c_2(\mathbf{A}_2)_{ijklpq}, \tag{41}$$

where $a_n (n = 1, \ldots, 5)$, d, $d_n (n = 1, 2, 3)$, b_n, and $c_n (n = 1, 2)$ are the higher-order material constants, and the tensor $\mathbf{S}_n (n = 1, \ldots, 5)$, $\mathbf{S}_3$, $\mathbf{D}_n (n = 1, 2, 3)$, $\mathbf{K}_1$ and $\mathbf{K}_2$, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the basic tensors provided in Appendix for the elasticity tensors F_{ijklpq}, F^h_{ijklpq}, F^d_{ijklpq}, F^s_{ijklpq}, and F^a_{ijklpq}, respectively.

Substituting Eq. (37) into Eq. (34), the constitutive relations for the higher-order metrics are written as

$$\begin{aligned}\tau_{ijk} = F_{ijklpq}\eta_{lpq} &= \frac{1}{4}a_1\left(\delta_{ij}\eta_{knn} + \delta_{ik}\eta_{jnn} + 2\delta_{jk}\eta_{nni}\right) \\ &+ \frac{1}{2}a_2\left(\delta_{ij}\eta_{nnk} + \delta_{ik}\eta_{nnj}\right) + a_3\delta_{jk}\eta_{inn} + a_4\eta_{ijk} \\ &+ \frac{1}{2}a_5\left(\eta_{kij} + \eta_{jki}\right),\end{aligned} \tag{42}$$

which is similar to the form presented by Mindlin (1964). Considering the relations,

$$\eta_{nni} = \frac{1}{3}\eta_{inn} + \varepsilon'_{ni,n}, \quad \eta_{ijk} = \frac{1}{3}\delta_{jk}\eta_{inn} + \varepsilon'_{jk,i}, \tag{43}$$

it can be recognized that the terms in Eq. (42) are coupled, that is, the five higher-order constants are not independent. The relationship between these five constants will be found by expressing the constitutive relations in terms of the components of higher-order metrics. From the expression of higher-order strain energy density in Eq. (35), the higher-order constitutive relations can be obtained as

$$\tau^h_{ijk} = F^h_{ijklpq}\eta^h_{lpq}, \quad \tau'_{ijk} = F^d_{ijklpq}\eta'_{lpq}. \tag{44}$$

Another form of higher-order constitutive relations can be obtained from Eq. (36), as follows

$$\tau^s_{ijk} = F^s_{ijklpq}\eta^s_{lpq}, \quad \tau^a_{ijk} = F^a_{ijklpq}\eta^a_{lpq}. \tag{45}$$

Applying the decomposing relations and the orthogonal properties of the higher-order components, the higher-order stress can also be expressed in terms of the above-described elasticity tensors as

$$\tau_{ijk} = \tau^s_{ijk} + \tau^a_{ijk} = \left(F^s_{ijklpq} + F^a_{ijklpq}\right)\eta_{lpq}, \tag{46}$$

$$\tau_{ijk} = \tau^h_{ijk} + \tau'_{ijk} = \left(F^h_{ijklpq} + F^d_{ijklpq}\right)\eta_{lpq}. \tag{47}$$

Thus, had the hydrostaic/deviatoric model in Eq. (44) and the symmetric/antisymmetric model in Eq. (45), together with the general five-constants one in Eq. (42), to represent a same isotropic strain gradient material, that is Eq. (42) equals Eqs. (46) and (47), the relationship between these sixth-order isotropic tensors is obtained as follows:

$$F_{ijklpq} = F^s_{ijklpq} + F^a_{ijklpq} = F^h_{ijklpq} + F^d_{ijklpq}. \tag{48}$$

By suing the relationships between these basic tensors provided in Appendix, we can obtain the relationships between the higher-order material constants for the strain gradient tensor as

$$a_1 = -\frac{2}{3}(a_2 + a_5),\ a_3 = \frac{2}{3}a_2 + \frac{1}{6}a_5, \tag{49}$$

for the hydrostatic and deviatoric components as

$$d = \frac{5}{9}d_1 + \frac{1}{3}d_2 + \frac{1}{18}d_3, \tag{50}$$

and for the symmetric and anti-symmetric parts as

$$b_1 = \frac{4}{5}c_1 - \frac{6}{5}(b_2 - c_2). \tag{51}$$

These relations show that any expressed form of the constitutive relations using either the strain gradient tensor or its components contains only three independent higher-order constants. We can conclude that there are only three independent higher-order material constants involved in the general isotropic strain gradient elasticity theory. This conclusion is different from that drawn by both Lam et al. (2003) and Fleck and Hutchinson (2001). In the theory by Lam et al., the symmetric character is enforced to the couple stress tensor, whereas the theory by Fleck and Hutchinson is plasticity theory only for incompressible materials. The demonstration procedure of independent higher-order constants as presented in this section can be similarly applied to the reformulation of the nonlinear second-gradient model proposed by Dell'Isola et al. (2009).

4.2. Constitutive relations

According to Eqs. (34), (42) and (49), the constitutive equations of isotropic strain gradient elasticity theories are rewritten as

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{nn} + 2\mu\varepsilon_{ij}, \tag{52}$$

and

$$\begin{aligned}\tau_{ijk} = &\frac{1}{2}a_2\left(\delta_{ij}\eta_{nnk} + \delta_{ik}\eta_{nnj}\right)\\ &-\frac{1}{6}(a_2 + a_5)\left(\delta_{ij}\eta_{kmm} + \delta_{ik}\eta_{jmm} + 2\delta_{jk}\eta_{nni}\right)\\ &+\frac{1}{6}(4a_2 + a_5)\delta_{jk}\eta_{inn} + a_4\eta_{ijk} + \frac{1}{2}a_5\left(\eta_{kij} + \eta_{jki}\right),\end{aligned} \tag{53}$$

with three independent higher-order elastic constants in addition to the traditional Lame constants. Further, in order to identify the specific corresponding relationship between solids deformation and its strain gradient components, constitutive relations in the form of higher-order strain and stress components are obtained from Eq. (53), according to the first set of higher-order metrics in Eqs. (14)–(24), as follows:

$$\tau^h_{ijk} = \left(\frac{5}{3}a_2 + a_4 + \frac{1}{6}a_5\right)\eta^h_{ijk}, \tag{54}$$

$$\tau^{(1)}_{ijk} = (a_4 + a_5)\eta^{(1)}_{ijk}, \tag{55}$$

$$\tau^{as}_{ijk} = \left(a_4 - \frac{1}{2}a_5\right)\eta^{as}_{ijk}, \tag{56}$$

$$\tau'^{(2)}_{ijk} = \left(\frac{5}{3}a_2 + a_4 + \frac{1}{6}a_5\right)\eta'^{(2)}_{ijk}. \tag{57}$$

Therefore, the strain energy density can be rewritten as

$$\begin{aligned}w &= \frac{1}{2}\sigma_{ij}\varepsilon_{ij} + \frac{1}{2}\tau^h_{ijk}\eta^h_{ijk} + \frac{1}{2}\tau^{(1)}_{ijk}\eta^{(1)}_{ijk} + \frac{1}{2}\tau^{as}_{ijk}\eta^{as}_{ijk} + \frac{1}{2}\tau'^{(2)}_{ijk}\eta'^{(2)}_{ijk}\\ &= \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + \frac{1}{2}\left(\frac{5}{3}a_2 + a_4 + \frac{1}{6}a_5\right)\eta^h_{ijk}\eta^h_{ijk}\\ &\quad + \frac{1}{2}(a_4 + a_5)\eta^{(1)}_{ijk}\eta^{(1)}_{ijk} + \frac{1}{2}\left(a_4 - \frac{1}{2}a_5\right)\eta^{as}_{ijk}\eta^{as}_{ijk}\\ &\quad + \frac{1}{2}\left(\frac{5}{3}a_2 + a_4 + \frac{1}{6}a_5\right)\eta'^{(2)}_{ijk}\eta'^{(2)}_{ijk}.\end{aligned} \tag{58}$$

The last two invariants of Eq. (58) are

$$\begin{aligned}&\eta^{as}_{ijk}\eta^{as}_{ijk} = \frac{2}{3}\chi^s_{ij}\chi^s_{ij} = \frac{2}{3}\chi'^s_{ij}\chi'^s_{ij} = \frac{1}{3}\left(\chi'_{ij}\chi'_{ij} + \chi'_{ij}\chi'_{ji}\right),\ \textit{and}\\ &\eta'^{(2)}_{ijk}\eta'^{(2)}_{ijk} = \frac{6}{5}\chi'^a_{ij}\chi'^a_{ij} = \frac{3}{5}\left(\chi'_{ij}\chi'_{ij} - \chi'_{ij}\chi'_{ji}\right).\end{aligned} \tag{59}$$

In order to evaluate the contribution of each strain gradient component intuitively, the constants are defined as,

$$\begin{aligned}&\frac{1}{2}\left(\frac{5}{3}a_2 + a_4 + \frac{1}{6}a_5\right) = 3\mu l_0^2,\ \frac{1}{2}(a_4 + a_5)\\ &\qquad = \mu l_1^2,\ \text{and}\ \frac{1}{2}\left(a_4 - \frac{1}{2}a_5\right) = 3\mu l_2^2,\end{aligned} \tag{60}$$

where $l_n (n = 0, 1, 2)$ are three length-scale parameters with the dimension of length. Therefore, the strain energy density is rewritten as

$$\begin{aligned}w = &\frac{1}{2}k\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon'_{ij}\varepsilon'_{ij} + \mu l_0^2\varepsilon_{nn,i}\varepsilon_{mm,i} + \mu l_1^2\eta^{(1)}_{ijk}\eta^{(1)}_{ijk}\\ &+ \mu\left(l_2^2 + \frac{9}{5}l_0^2\right)\chi'_{ij}\chi'_{ij} + \mu\left(l_2^2 - \frac{9}{5}l_0^2\right)\chi'_{ij}\chi'_{ji},\end{aligned} \tag{61}$$

where k and μ are the bulk and shear modulus, respectively. The constitutive relations of the general strain gradient elasticity are obtained from Eq. (61), as follows:

$$\sigma_{ij} = \frac{\partial w}{\partial\varepsilon_{ij}} = k\delta_{ij}\varepsilon_{nn} + 2\mu\varepsilon'_{ij}, \tag{62}$$

$$p_i = \frac{\partial w}{\partial\varepsilon_{nn,i}} = 2\mu l_0^2\varepsilon_{nn,i}, \tag{63}$$

$$\tau^{(1)}_{ijk} = \frac{\partial w}{\partial\eta^{(1)}_{ijk}} = 2\mu l_1^2\eta^{(1)}_{ijk}, \tag{64}$$

$$m'_{ij} = \frac{\partial w}{\partial\chi'_{ij}} = 2\mu\left(l_2^2 + \frac{9}{5}l_0^2\right)\chi'_{ij} + 2\mu\left(l_2^2 - \frac{9}{5}l_0^2\right)\chi'_{ji}, \tag{65}$$

where p_i, $\tau^{(1)}_{ijk}$, and m'_{ij} are work-conjugates to $\varepsilon_{nn,\,i}$, $\eta^{(1)}_{ijk}$, and χ'_{ij}, respectively.

For the incompressible material in which the hydrostatic deformation vanishes, Eq. (61) simplifies to

$$\begin{aligned}w = &\mu\varepsilon'_{ij}\varepsilon'_{ij} + \mu l_1^2\eta^{(1)}_{ijk}\eta^{(1)}_{ijk} + \mu\left(l_2^2 + \frac{9}{5}l_0^2\right)\chi'_{ij}\chi'_{ij}\\ &+ \mu\left(l_2^2 - \frac{9}{5}l_0^2\right)\chi'_{ij}\chi'_{ji}.\end{aligned} \tag{66}$$

The number of length-scale parameters is equal to that of the strain gradient plasticity by Fleck and Hutchinson (1997). In the strain gradient plasticity, the effect of the term $\chi'_{ij}\chi'_{ji}$ in the effective plastic strain is usually excluded, because no example has been identified yet for which this invariant plays a particularly important role

(Begley and Hutchinson, 1998; Fleck and Hutchinson, 2001). Here, if we exclude any dependence on this invariant in the deformation energy density function for the elasticity theory by assuming $l_2 = 3/\sqrt{5}\, l_0$, the number of length parameters will further reduce to two from three, resulting in the modification of the strain energy density (42) as

$$w = \frac{1}{2}k\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon'_{ij}\varepsilon'_{ij} + \mu l_0^2\varepsilon_{nn,i}\varepsilon_{kk,i} + \mu l_1^2\eta_{ijk}^{(1)}\eta_{ijk}^{(1)} + \frac{18}{5}\mu l_0^2\chi'_{ij}\chi'_{ij}. \tag{67}$$

Thus, experimental data from two different types of micro-scale test are able to independently determine the length parameters l_0 and l_1, such as torsion and bending tests.

4.3. Governing equations and boundary conditions

According to Eqs. (13)–(18), the strain energy density in Eq. (61) can also be rewritten as

$$\begin{aligned} w = {} & \frac{1}{2}k\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon'_{ij}\varepsilon'_{ij} + \mu\left(\frac{9}{5}l_0^2 - \frac{4}{15}l_1^2 - l_2^2\right)\eta_{iik}\eta_{jjk} \\ & -\mu\left(\frac{6}{5}l_0^2 + \frac{4}{15}l_1^2 - 2l_2^2\right)\eta_{kjj}\eta_{iik} + \mu\left(\frac{6}{5}l_0^2 - \frac{1}{15}l_1^2 - l_2^2\right)\eta_{kii}\eta_{kjj} \\ & +\mu\left(\frac{1}{3}l_1^2 + 2l_2^2\right)\eta_{ijk}\eta_{ijk} + \mu\left(\frac{2}{3}l_1^2 - 2l_2^2\right)\eta_{kij}\eta_{ijk}, \end{aligned} \tag{68}$$

which is similar to that of Mindlin (1965), but the independent higher-order constants are three in the current theory rather than five. Then, the constitutive relations are derived as

$$\sigma_{ij} = k\delta_{ij}\varepsilon_{nn} + 2\mu\varepsilon'_{ij}, \tag{69}$$

$$\begin{aligned} \tau_{ijk} = {} & \mu\left(\frac{9}{5}l_0^2 - \frac{4}{15}l_1^2 - l_2^2\right)\left(\delta_{ij}\eta_{nnk} + \delta_{ik}\eta_{nnj}\right) \\ & -\mu\left(\frac{3}{5}l_0^2 + \frac{2}{15}l_1^2 - l_2^2\right)\left(\delta_{ij}\eta_{kmm} + \delta_{ik}\eta_{jmm} + 2\delta_{jk}\eta_{nni}\right) \\ & +\mu\left(\frac{12}{5}l_0^2 - \frac{2}{15}l_1^2 - 2l_2^2\right)\delta_{jk}\eta_{inn} + \mu\left(\frac{2}{3}l_1^2 + 4l_2^2\right)\eta_{ijk} \\ & +\mu\left(\frac{2}{3}l_1^2 - 2l_2^2\right)\left(\eta_{kij} + \eta_{jki}\right). \end{aligned} \tag{70}$$

In addition, the equilibrium equation and boundary conditions have been provided in Eqs. (6)–(10). Although the equilibrium equation and boundary conditions listed in Eqs. (6)–(10) are simple and easy to apply, the equilibrium equation and boundary conditions in the form of higher-order stress components are also essential for the sake of identifying the contribution of each component. Applying the principle of the virtual work,

$$\begin{aligned} & \int_V \left(\sigma_{ij}\delta\varepsilon_{ij} + p_i\delta\varepsilon_{,i} + \tau_{ijk}^{(1)}\delta\eta_{ijk}^{(1)} + m'_{ij}\delta\chi'_{ij}\right)\mathrm{d}V = \int_V b_k\delta u_k\mathrm{d}V \\ & \quad + \int_S \left(\bar{t}_k\delta u_k + \bar{\mu}_k(\delta_{kl} - n_kn_l)\delta\theta_l + \bar{r}\delta\varepsilon_N\right)\mathrm{d}S + \oint_C \bar{f}_k\delta u_k\mathrm{d}C, \end{aligned} \tag{71}$$

the equilibrium equation in the volume V is given as

$$\sigma_{ik,i} - p_{i,ik} - \tau_{ijk,ji}^{(1)} - \frac{1}{2}e_{ilk}m'_{ij,lj} + \frac{1}{3}e_{ilj}m'_{ij,lk} + b_k = 0, \tag{72}$$

as well as, the boundary conditions on the surface S are

$$\begin{aligned} \bar{t}_k = {} & n_i[\sigma_{ik} - p_{n,n}\delta_{ik} - \tau_{ijk,j}^{(1)} - \frac{1}{2}e_{lik}m'_{lj,j} + \frac{1}{3}e_{ljs}m'_{ls,j}\delta_{ik}] \\ & +(D_pn_p)\left(n_kn_ip_i + n_in_j\tau_{ijk}^{(1)} + n_i\tau_{ijs}^{(1)}n_jn_kn_s - \frac{1}{3}e_{ijs}n_kn_jm'_{is}\right) \\ & -D_s\left(\frac{1}{2}e_{jsk}n_in_lm'_{il}n_j + n_i\tau_{sik}^{(1)} + n_i\tau_{ijs}^{(1)}n_jn_k\right) \\ & -D_k\left(n_ip_i - \frac{1}{3}e_{ijs}m'_{is}n_j\right) \text{ or } \bar{u}_k = u_k, \end{aligned} \tag{73}$$

$$\bar{\mu}_k = n_lm'_{kl} - 2e_{slk}n_i\tau_{ijs}^{(1)}n_jn_l \text{ or } \bar{\theta}_k = (\delta_{ki} - n_kn_i)\theta_i, \tag{74}$$

$$\bar{r} = n_ip_i + n_i\tau_{ijk}^{(1)}n_jn_k - \frac{1}{3}e_{isj}n_sm'_{ij} \text{ or } \bar{\varepsilon}_N = n_in_j\varepsilon_{ij}, \tag{75}$$

and along the edge C of the piece-wise smooth surface is

$$\begin{aligned} \bar{f}_k = {} & \Big[k_k(n_ip_i) + k_jn_i\tau_{jik}^{(1)} + \frac{1}{2}e_{jsk}\left(n_im'_{il}n_l\right)k_sn_j + n_in_jk_l\tau_{ijl}^{(1)}n_k \\ & -\frac{1}{3}k_k\left(e_{ijl}n_jm'_{il}\right)\Big] \text{ or } \bar{u}_k = u_k, \end{aligned} \tag{76}$$

where θ_i is the rotation vector. $\bar{\theta}_k$ and $\bar{\varepsilon}_N$are the given rotation tangential to the surface and the given normal strain, respectively. New equilibrium and boundary conditions in terms of the independent metrics are now established for the present theory. These equations can directly reduce to the case of incompressible materials when the terms associated with the hydrostatic strain vanish.

5. Solutions to simple problems

The general isotropic strain gradient elasticity theory in the form of strain gradient components offers a good opportunity to identify the contribution of each strain gradient component. In this section, three basic problems are considered within the framework of the present theory with three length-scale parameters and the simple theory with single parameter. The results of present three-parameter theory will reveal which strain gradient component enters the corresponding problem and controls its size effect. Comparison of solutions from the three-parameter and one-parameter versions allows one to draw conclusions about the necessary of including multiple material length parameters in strain gradient theory. For the simple theory without surface energy by Vardoulakis et al. (1996), the strain energy density has the form as

$$w_s = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + l^2\left(\frac{1}{2}\lambda\eta_{inn}\eta_{ikk} + \mu\eta_{ijk}\eta_{ijk}\right), \tag{77}$$

which is a special case of the Mindlin theory. Obviously, the strain energy density (77) contains only one length-scale parameter l with the dimension of length.

5.1. Torsion of bar

Consider a cylindrical bar with constant radius R, whose axis is assumed to be the z-axis. When the bar twists, the displacement components, same as those in classical theory, are expressed as

$$u_1 = -\theta yz, \quad u_2 = \theta xz, \quad u_3 = 0, \tag{78}$$

where θ is the constant angle of twist per unit length. The derived strain is identical to that in the classical theory. The hydrostatic strain equals zero, and the non-zero deviatoric strains are

$$\varepsilon'_{13} = \varepsilon'_{31} = -\frac{1}{2}\theta y,\ \varepsilon'_{23} = \varepsilon'_{32} = \frac{1}{2}\theta x. \tag{79}$$

For the three-parameter version, the dilatation gradient and the deviatoric stretch gradient vanish, and the deviatoric rotation gradient components, the only non-vanishing strain gradient quantities, are

$$\chi'_{11} = -\frac{1}{2}\theta,\ \chi'_{22} = -\frac{1}{2}\theta, \text{ and } \chi'_{33} = \theta. \tag{80}$$

From Eq. (61), the strain energy density can be obtained as

$$w = \frac{1}{2}\mu\theta^2\left(R^2 + 6l_2^2\right). \tag{81}$$

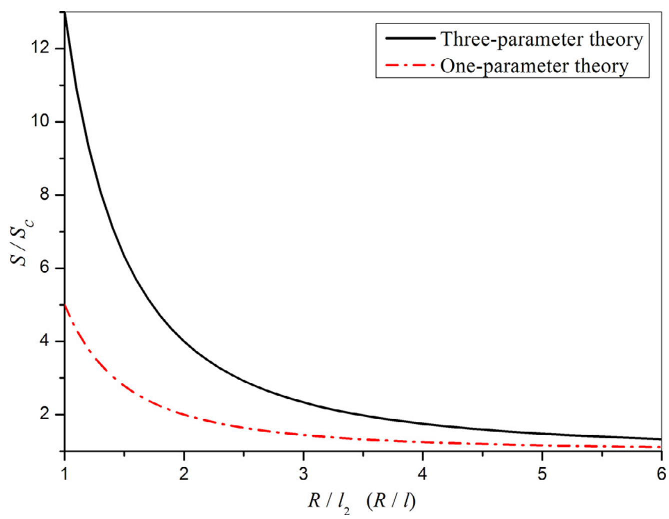

Fig. 1. Size effect of the normalized torsion rigidity.

On the basis of the work-energy principle,

$$\int_0^L \int_A w \mathrm{d}A\mathrm{d}z = \frac{1}{2} T\theta L, \tag{82}$$

the twist angle is derived as

$$\theta = \frac{T}{\mu\left(I_p + 6Al_2^2\right)}, \tag{83}$$

where T denotes the torque, L denotes the length of the bar, A denotes the area of cross section, and I_p denotes the polar moment of inertia of the cross section. Therefore, the torsion rigidity is obtained from the above relation as

$$S = \mu\left(I_p + 6Al_2^2\right). \tag{84}$$

For the one-parameter version, the torsion rigidity is given by

$$S = \mu\left(I_p + 2Al^2\right). \tag{85}$$

Considering the torsion rigidity for the classical theory, $S_c = \mu I_p$, the torsion rigidities for both versions can be normalized as

$$\frac{S}{S_c} = 1 + 12\frac{l_*^2}{R^2}, \tag{86}$$

where l_* depends on the formulation according to $l_*^2 = l_2^2$ for the three-parameter version and $l_*^2 = 1/3\, l^2$ for the one-parameter version.

From the formulation of the three-parameter version, one knows that only the length parameter l_2 determines the size effect in bar torsion, and the other length parameters l_0 and l_1 do not enter this problem. The contribution of the three-parameter version to the size effect of the torsion rigidity is $6\mu Al_2^2$, while the simple version has $2\mu Al^2$ as its contribution to the torsion rigidity. The results from the two theories in the form of normalized torsion rigidities as a function of the ratio of radius to length parameter are compared in Fig. 1. It is seen that the size-dependencies of normalized torsion rigidities from gradient effects predicted in the two versions are obviously different. Identical gradient effects from the two versions would be expected when $l^2 = 3l_2^2$. Nevertheless, the factor is strongly problem-dependent, which will be seen in the following shear and pure bending problems.

5.2. Shearing of fixed-end layers

Consider a block of width b, length L and height h undergoing a shear deformation, as shown in Fig. 2. Assume b and L are much larger than h. The Cartesian coordinate system shown in the figure is adopted in the formulation below. The only non-vanishing x-component $u(y)$ of the displacement vector is induced by a shear force P acting on the top surface $y = h$. The hydrostatic strain equals zero, and the non-zero deviatoric strains are

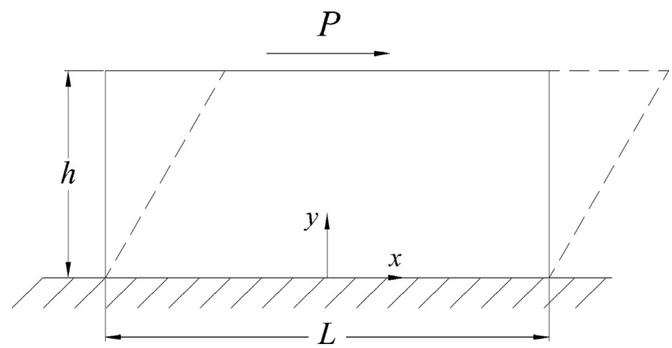

Fig. 2. Simple shear problem.

$$\varepsilon'_{xy} = \varepsilon'_{yx} = \frac{1}{2}\frac{\mathrm{d}u}{\mathrm{d}y}. \tag{87}$$

In the three-parameter version, the dilatation gradient vanishes. The non-zero strain gradient quantities are the deviatoric stretch gradient and the deviatoric rotation gradient, which are given as

$$\eta_{111}^{(1)} = -\frac{1}{5}\frac{\mathrm{d}^2u}{\mathrm{d}y^2},\ \eta_{221}^{(1)} = \eta_{212}^{(1)} = \eta_{122}^{(1)} = \frac{4}{15}\frac{\mathrm{d}^2u}{\mathrm{d}y^2},\ \eta_{331}^{(1)} = \eta_{313}^{(1)} = \eta_{133}^{(1)} = -\frac{1}{15}\frac{\mathrm{d}^2u}{\mathrm{d}y^2}, \tag{88}$$

$$\chi'_{32} = -\frac{1}{2}\frac{\mathrm{d}^2u}{\mathrm{d}y^2}. \tag{89}$$

The work-conjugated stress and higher-order stress quantities obtained from Eqs. (62)–(65) are listed as

$$\sigma_{xy} = \sigma_{yx} = \mu\frac{\mathrm{d}u}{\mathrm{d}y}, \tag{90}$$

$$m'_{32} = -\mu\left(l_2^2 + \frac{9}{5}l_0^2\right)\frac{\mathrm{d}^2u}{\mathrm{d}y^2},\ m'_{23} = -\mu\left(l_2^2 - \frac{9}{5}l_0^2\right)\frac{\mathrm{d}^2u}{\mathrm{d}y^2}, \tag{91}$$

$$\tau_{111}^{(1)} = -\frac{2}{5}\mu l_1^2\frac{\mathrm{d}^2u}{\mathrm{d}y^2},\ \tau_{221}^{(1)} = \tau_{122}^{(1)} = \tau_{212}^{(1)} = \frac{8}{15}\mu l_1^2\frac{\mathrm{d}^2u}{\mathrm{d}y^2},\ \tau_{331}^{(1)} = \tau_{313}^{(1)} = \tau_{133}^{(1)} = -\frac{2}{15}\mu l_1^2\frac{\mathrm{d}^2u}{\mathrm{d}y^2}. \tag{92}$$

Substituting Eqs. (90)–(92) into Eq. (72) leads to the governing equation,

$$g^2\frac{\mathrm{d}^4u}{\mathrm{d}y^4} - \frac{\mathrm{d}^2u}{\mathrm{d}y^2} = 0, \tag{93}$$

in which $g^2 = \frac{8}{15}l_1^2 + \frac{1}{2}l_2^2 + \frac{9}{10}l_0^2$. According to the boundary conditions of the current problem

$$u(0) = 0,\ \varepsilon_{xy}(0) = 0,\ \frac{\mathrm{d}u(h)}{\mathrm{d}y} - g^2\frac{\mathrm{d}^3u(h)}{\mathrm{d}y^3} = \frac{P}{\mu bL},\ \frac{\mathrm{d}^2u(h)}{\mathrm{d}y^2} = 0, \tag{94}$$

the solution of Eq. (93) can be readily obtained as

$$u(y) = \frac{P}{\mu bL}[y + g\frac{\sinh[(1-\xi)\frac{h}{g}]}{\cosh\left(\frac{h}{g}\right)} - g\tanh\left(\frac{h}{g}\right)], \tag{95}$$

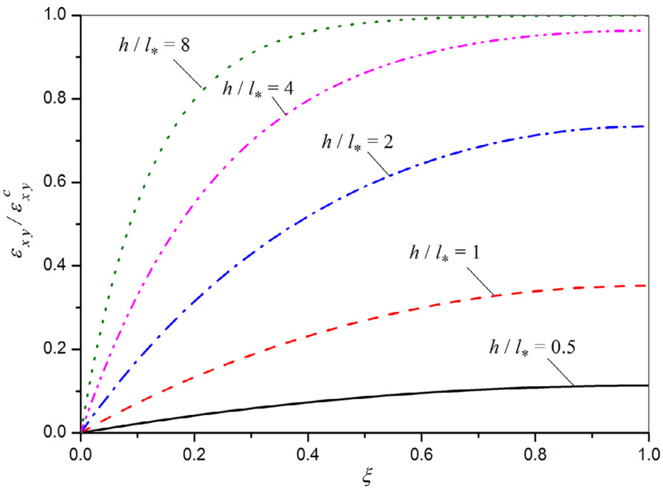

Fig. 3. Distribution of normalized shear strain with different characteristic size.

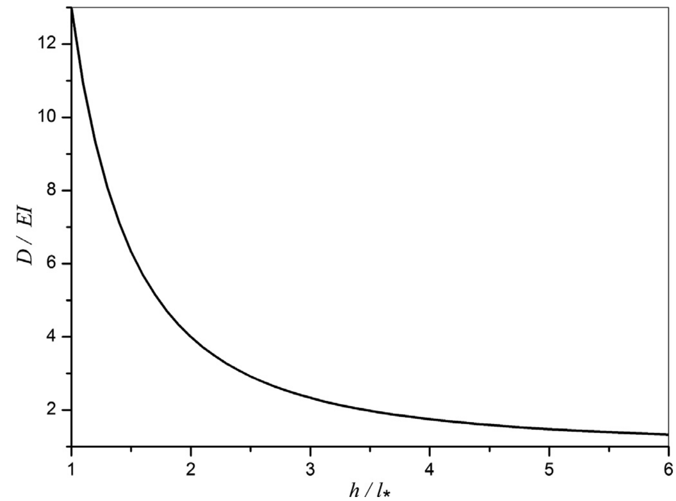

Fig. 4. Size effect of normalized bending rigidity.

where the normalized coordinate $\xi = y/h$. The non-vanishing strain, hence, is found to be

$$\varepsilon_{xy} = \frac{1}{2}\frac{P}{\mu bL}\left(1 - \frac{\cosh\left[(1-\xi)\frac{h}{g}\right]}{\cosh\left(\frac{h}{g}\right)}\right), \tag{96}$$

and the corresponding strain for the one-parameter version is

$$\varepsilon_{xy} = \frac{1}{2}\frac{P}{\mu bL}\left(1 - \frac{\cosh\left[(1-\xi)\frac{h}{l}\right]}{\cosh\left(\frac{h}{l}\right)}\right). \tag{97}$$

Considering the non-vanishing strain in the classical theory, $\varepsilon_{xy}^c = \frac{1}{2}\frac{P}{\mu bL}$, the non-vanishing strains in the two versions can be normalized as

$$\frac{\varepsilon_{xy}}{\varepsilon_{xy}^c} = 1 - \frac{\cosh[(1-\xi)\frac{h}{l_*}]}{\cosh\left(\frac{h}{l_*}\right)}, \tag{98}$$

where l_* depends on the formulation according to $l_* = g$ for the three-parameter version and $l_* = l$ for the one-parameter version.

The distribution of the normalized shear strain along the height of the block in Eq. (98) is shown in Fig. 3 for various values of h/l_*. The size-dependent normalized shear strain is seen in the shear problem. Size effects of shear strain predicted in the two formulations are expected to be identical when $l^2 = \frac{8}{15}l_1^2 + \frac{1}{2}l_2^2 + \frac{9}{10}l_0^2$. Thus, the corresponding factor is 1/2 if $l_0 = l_1 = 0$, which is different from the factor in the torsion problem. One knows that the equivalent relations in the torsion and shear problems are strongly problem-dependent rather than only the deformation of materials. If each of the two formulations were to be calibrated by the torsion problem such that $l^2 = 3l_2^2$, then at least one of the versions must clearly be significantly erroneous in predicting shear problem. This strong problem dependence of the size effect is the essence underlying the necessity of the multiple-parameter theory to predict the size effect in microstructures. Here, we arrive at the same conclusion as in the plastic deformation theory (Fleck and Hutchinson, 2001).

5.3. Pure bending of thin beams

Consider a rectangular beam with height h, width b and length L. The Cartesian coordinate system is used. We assume the x-axis to coincide with the centerline of the beam and the other axes parallel to the sides of the cross section. The displacement components are

$$u_1 = \frac{1}{R}xz, \quad u_2 = -\frac{v}{R}yz, \quad u_3 = \frac{v}{2R}\left(y^2 - z^2\right) - \frac{1}{2R}x^2, \tag{99}$$

where R denotes the radius of curvature of the central axis of the beam after bending in the xz-plane. Then, the strains can be written as

$$\varepsilon_{11} = \frac{z}{R},\ \varepsilon_{22} = -\frac{v}{R}z,\ \varepsilon_{33} = -\frac{v}{R}z. \tag{100}$$

For the three-parameter version, the high-order deformation quantities are given as

$$\varepsilon_{nn,3} = \frac{1}{R}(1-2v), \tag{101}$$

$$\chi'_{12} = \frac{1}{3}\frac{1+v}{R},\ \chi'_{21} = \frac{2}{3}\frac{1+v}{R}, \tag{102}$$

$$\eta_{333}^{(1)} = -\frac{1}{5}\frac{1}{R}(1+v), \tag{103}$$

$$\eta_{113}^{(1)} = \eta_{131}^{(1)} = \eta_{311}^{(1)} = \frac{4}{15}\frac{1}{R}(1+v),\ \eta_{223}^{(1)} = \eta_{232}^{(1)} = \eta_{322}^{(1)} = -\frac{1}{15}\frac{1}{R}(1+v). \tag{104}$$

From Eq. (61), the strain energy density can be obtained as

$$w = \frac{1}{2}\frac{1}{R^2}\left\{Ez^2 + 2\mu\left[l_0^2(1-2v)^2 + \left(\frac{1}{5}l_0^2 + \frac{4}{15}l_1^2 + l_2^2\right)(1+v)^2\right]\right\}. \tag{105}$$

On the basis of the work-energy principle,

$$\int_0^L\int_A w\mathrm{d}A\mathrm{d}x = \frac{1}{2}M\frac{L}{R}, \tag{106}$$

the curvature of the central axis of the beam after bending is obtained as

$$\frac{1}{R} = \frac{M}{D}, \tag{107}$$

where M denotes the bending moment and D denotes the bending rigidity expressed as

$$D = EI + 2\mu A\left[l_0^2(1-2v)^2 + \left(\frac{1}{5}l_0^2 + \frac{4}{15}l_1^2 + l_2^2\right)(1+v)^2\right], \tag{108}$$

where E is the Young's modulus, v is the Poisson ratio, and I is the moment of inertia. The corresponding bending rigidity for the one-parameter version is

$$D = EI + EAl^2. \tag{109}$$

The bending rigidities, including gradient effects, in the two versions are normalized using the bending rigidity of the classical theory, *EI*, as follows

$$\frac{D}{EI} = 1 + 12\frac{l_*^2}{h^2}, \tag{110}$$

where l_*^2 depends on the formulation according to $l_*^2 = l_0^2\frac{(1-2\nu)^2}{1+\nu} + (\frac{1}{5}l_0^2 + \frac{4}{15}l_1^2 + l_2^2)(1+\nu)$ for the three-parameter version and $l_*^2 = l^2$ for the one-parameter version. The size-dependent bending rigidity of pure bending of a thin beam with rectangular cross-section is shown in Fig. 4. It is seen that only if the parameters in the three- and one-parameter versions hold the equivalent relation $l^2 = l_0^2\frac{(1-2\nu)^2}{1+\nu} + (\frac{1}{5}l_0^2 + \frac{4}{15}l_1^2 + l_2^2)(1+\nu)$, the two versions will predict the same size effects for the pure bending problem. The equivalent relation is different from the bar torsion and shear problem. The different equivalent relation between the three- and one-parameter versions in the pure bending problem further shows the necessity of using the multiple-parameter theory to predict size effects.

6. Conclusion

By applying two sets of orthogonal decompositions of the strain gradient tensor, the general isotropic strain gradient elasticity theory with only three independent length-scale parameters is reformulated. In the reformulated frame, the total deformation energy density is a function of four parts: the symmetric strain tensor, the dilatation gradient vector, the deviatoric stretch gradient tensor, and the deviatoric curvature tensor. The independent strain gradient parts and the corresponding work-conjugated stress tensors are defined. After strict derivations, the constitutive equations, equilibrium equation and boundary conditions are obtained subsequently. In addition, the deformation energy density, equilibrium equations, and boundary conditions can directly reduce to the case of incompressible materials by assuming the dilatation gradient to be zero.

Three simple examples are studied based on the present strain gradient elasticity theory. The results reveal that the dilatation gradient, the deviatoric stretch gradient, and the deviatoric rotation gradient control different higher-order deformation in isotropic solids. The nonvanishing higher-order deformation in torsion is the deviatoric rotation gradient. The dilatation gradient vanishes in the shear problem. All the higher-order deformations enter the pure bending problem. By comparing the present solutions of three examples with those of the simple strain gradient theory containing only one higher-order length-scale parameter, the necessity of including more than one length-scale parameters in the strain gradient theory to consider size effects in the micron scale has been explained. The present strain gradient theory can provide effective descriptions of size-dependent behaviors in wide micron-scale isotropic elastic problems.

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Appendix. The basic tensors of sixth-order isotropic tensors

In the higher-order elasticity theory, both the theories by Mindlin and Eshel (1968) and Dell'Isola et al. (2009), the constitutive equations for linear isotropic materials are a linear relation between two third-order tensors, denoted by a_{ijk} and b_{ijk}, which is expressed as

$$a_{ijk} = A_{ijklpq}b_{lpq}, \tag{A.1}$$

where A_{ijklpq} is the sixth-order elasticity tensor. The property required by isotropic materials implies the elasticity tensor to conform

$$A_{ijklpq} = A_{hmnrst}Q_{hi}Q_{mj}Q_{nk}Q_{rl}Q_{ps}Q_{tq}, \tag{A.2}$$

for every orthogonal transformation Q_{ij}. The tensor A_{ijklpq} can be read as a linear combination of sixth-order isotropic components $\mathbf{T}_n(n = 1...15)$, which are called as basic tensors, given by (Monchiet and Bonnet, 2011; Suiker and Chang, 2000)

$$\begin{aligned}
(\mathbf{T}_1)_{ijklpq} &= \delta_{ij}\delta_{kl}\delta_{pq}, (\mathbf{T}_2)_{ijklpq} = \delta_{ij}\delta_{kp}\delta_{lq}, (\mathbf{T}_3)_{ijklpq} = \delta_{ij}\delta_{kq}\delta_{lp},\\
(\mathbf{T}_4)_{ijklpq} &= \delta_{ik}\delta_{jl}\delta_{pq}, (\mathbf{T}_5)_{ijklpq} = \delta_{ik}\delta_{jp}\delta_{lq}, (\mathbf{T}_6)_{ijklpq} = \delta_{ik}\delta_{jq}\delta_{lp},\\
(\mathbf{T}_7)_{ijklpq} &= \delta_{il}\delta_{jk}\delta_{pq}, (\mathbf{T}_8)_{ijklpq} = \delta_{il}\delta_{jp}\delta_{kq}, (\mathbf{T}_9)_{ijklpq} = \delta_{il}\delta_{jq}\delta_{kp},\\
(\mathbf{T}_{10})_{ijklpq} &= \delta_{ip}\delta_{jk}\delta_{lq}, (\mathbf{T}_{11})_{ijklpq} = \delta_{ip}\delta_{jl}\delta_{kq}, (\mathbf{T}_{12})_{ijklpq} = \delta_{ip}\delta_{jq}\delta_{kl},\\
(\mathbf{T}_{13})_{ijklpq} &= \delta_{iq}\delta_{jk}\delta_{lp}, (\mathbf{T}_{14})_{ijklpq} = \delta_{iq}\delta_{jl}\delta_{kp}, (\mathbf{T}_{15})_{ijklpq} = \delta_{iq}\delta_{jp}\delta_{kl}.
\end{aligned} \tag{A.3}$$

If the sixth-order tensor A_{ijklpq} satisfies some special symmetry, then its basic tensors can be simplified. For a tensor having the major symmetries, $A_{ijklpq} = A_{lpqijk}$, only 11 basic tensors denoted $\mathbf{M}_n(n = 1...11)$ are needed. They are given by

$$\begin{aligned}
\mathbf{M}_1 &= \frac{1}{2}(\mathbf{T}_1 + \mathbf{T}_{13}), \mathbf{M}_2 = \frac{1}{2}(\mathbf{T}_2 + \mathbf{T}_6), \mathbf{M}_3 = \mathbf{T}_3, \mathbf{M}_4\\
&= \frac{1}{2}(\mathbf{T}_4 + \mathbf{T}_{10}), \mathbf{M}_5 = \mathbf{T}_5\mathbf{M}_6 = \mathbf{T}_7, \mathbf{M}_7 = \mathbf{T}_8, \mathbf{M}_8 = \mathbf{T}_9,\\
\mathbf{M}_9 &= \mathbf{T}_{11}, \mathbf{M}_{10} = \frac{1}{2}(\mathbf{T}_{12} + \mathbf{T}_{14}), \mathbf{M}_{11} = \mathbf{T}_{15}.
\end{aligned} \tag{A.4}$$

Furthermore, if a tensor has the additional minor symmetries, $A_{ijklpq} = A_{ikjlpq} = A_{ijklqp}$, in addition to the major symmetries, the number of basic tensors reduces from 11 to five. These basic tensors are

$$\begin{aligned}
\mathbf{S}_1 &= \frac{1}{4}(\mathbf{T}_1 + \mathbf{T}_{10} + \mathbf{T}_{13} + \mathbf{T}_4), \mathbf{S}_2 = \frac{1}{4}(\mathbf{T}_2 + \mathbf{T}_3 + \mathbf{T}_5 + \mathbf{T}_6)\\
\mathbf{S}_3 &= \mathbf{T}_7, \mathbf{S}_4 = \frac{1}{2}(\mathbf{T}_8 + \mathbf{T}_9), \mathbf{S}_5 = \frac{1}{4}(\mathbf{T}_{11} + \mathbf{T}_{12} + \mathbf{T}_{14} + \mathbf{T}_{15}).
\end{aligned} \tag{A.5}$$

However, a case of the minor symmetries depending on a Kronecker symbol needs only the basic tensor $\mathbf{S}_3$; whereas, for a case that the components contracting two minor symmetric indices vanish, only three basic tensors are needed, given as

$$\mathbf{D}_1 = \mathbf{S}_2 - \frac{2}{3}\mathbf{S}_1 + \frac{1}{9}\mathbf{S}_3, \quad \mathbf{D}_2 = \mathbf{S}_4 - \frac{1}{3}\mathbf{S}_3, \quad \mathbf{D}_3 = \frac{1}{9}\mathbf{S}_3 + \mathbf{S}_5 - \frac{2}{3}\mathbf{S}_1. \tag{A.6}$$

When a tensor has either the additional symmetric properties $A_{ijklpq} = A_{kijlpq} = A_{jkilpq}$, or the additional anti-symmetric properties $A_{ijklpq} + A_{kijlpq} + A_{jkilpq} = 0$, in addition to the major and minor symmetries, only two basic tensors are needed. They are

$$\begin{aligned}
\mathbf{K}_1 &= \frac{1}{9}(4\mathbf{S}_1 + \mathbf{S}_3 + 4\mathbf{S}_2)\\
&= \frac{1}{9}(\mathbf{T}_1 + \mathbf{T}_{10} + \mathbf{T}_{13} + \mathbf{T}_4 + \mathbf{T}_7 + \mathbf{T}_2 + \mathbf{T}_3 + \mathbf{T}_5 + \mathbf{T}_6),
\end{aligned} \tag{A.7}$$

and

$$\begin{aligned}
\mathbf{K}_2 &= \frac{1}{3}(2\mathbf{S}_4 + 4\mathbf{S}_5)\\
&= \frac{1}{3}(\mathbf{T}_8 + \mathbf{T}_9 + \mathbf{T}_{11} + \mathbf{T}_{12} + \mathbf{T}_{14} + \mathbf{T}_{15}),
\end{aligned} \tag{A.8}$$

for the additional symmetric conditions, and

$$\begin{aligned}
\mathbf{A}_1 &= \frac{4}{9}(\mathbf{S}_3 + \mathbf{S}_2 - 2\mathbf{S}_1)\\
&= \frac{1}{9}(4\mathbf{T}_7 + \mathbf{T}_2 + \mathbf{T}_5 + \mathbf{T}_3 + \mathbf{T}_6 - 2\mathbf{T}_1 - 2\mathbf{T}_4 - 2\mathbf{T}_{10} - 2\mathbf{T}_{13}),
\end{aligned} \tag{A.9}$$

and

$$\mathbf{A}_2 = \frac{4}{3}(\mathbf{S}_4 - \mathbf{S}_5) = \frac{1}{3}(2\mathbf{T}_8 + 2\mathbf{T}_9 - \mathbf{T}_{11} - \mathbf{T}_{12} - \mathbf{T}_{14} - \mathbf{T}_{15}) \tag{A.10}$$

for the additional anti-symmetric conditions. Finally, these sixth-order tensors with special symmetry can be read as a linear combination of their basic tensors.

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