THE SUBCONVEXITY PROBLEM ON HIGHER RANK GROUPS

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Abstract. In this note we discuss an approach for the subconvexity problem on higher rank groups.

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1. Introduction

The purpose of this lecture series is to explain a recent approach on the subconvexity bound for higher rank groups. To achieve this goal we shall also explain some basic results for automorphic forms and automorphic representations.

The subconvexity bound means an upper bound of following shape

$$L(\Pi, 1/2) \ll C(\Pi)^{1/4-\delta}.$$

Here $L(\Pi, s)$ is the so-called L-function associated to a number-theoretical object Π, which will be automorphic representation or product of automorphic representation for us. $L(\Pi, s)$ has analytic continuation and functional equation as $\zeta(s)$. For example, $L(\Pi, 1/2)$ can be the L-function for a modular form is Π is associated to one.

$C(\Pi)$ is the so-called **analytic conductor** measuring the complexity of Π; it also appears in the functional equation for $L(\Pi, s)$.

$1/4$ in the exponent corresponding to the convexity bound, which can be proven using only functional equation and complex analysis. The goal is to obtain a subconvexity bound with $\delta > 0$, using number theoretic tools.

We shall mainly follow the strategy of [6], taking some simplifications along the way. We shall also start with some basics, but we will often skip the details or only give an intuitive explanation.

2. Adele,idele

Most of this section can be found in a course in algebraic number theory or class field theory. We shall focus on the rational field $\mathbb{Q}$. Most discussions carry over to general number fields.

2.1. **Local fields.**

There are two types of local fields associated to $\mathbb{Q}$:

(1) $\mathbb{R}$, the completion of $\mathbb{Q}$ with respect to usual absolute value $|x|_\infty := |x|$;
(2) p-adic fields $\mathbb{Q}_p$, the completion of $\mathbb{Q}$ with respect to p-adic norm $|x|_p = p^{-v_p(x)}$, where $v_p(x)$ is the exponent for p when we do a prime factorization for x.

We call ∞ the archimedean place of $\mathbb{Q}$ and any p the p-adic place of $\mathbb{Q}$. For uniformity we may use v to denote a place of $\mathbb{Q}$.

Any number in $\mathbb{Q}_p$ can be written as

$$x = \sum_{i \geq i_0} a_i p^i$$

for $a_i = 0, 1, \cdots, p-1$, $a_{i_0} \neq 0$. Then $v_p(x) = i_0$ and $|x|_p = p^{-i_0}$. We denote the ring of integers in $\mathbb{Q}_p$ by $\mathbb{Z}_p$ or $\mathcal{O} = \{x \in \mathbb{Q}_p, v_p(x) \geq 0\}$, with units $\mathbb{Z}_p^\times$ or $\mathcal{O}^\times$. Its unique maximal/prime ideal is $\mathfrak{p} = p\mathcal{O}$.

We consider the topology on $\mathbb{Q}_p$ so that the p-adic norm $|\cdot|_p$ is a continuous map. A very important feature for p-adic fields is that this topology is totally disconnected, meaning that its open subsets are also closed. For example, let $\mathbb{F} = \mathbb{Q}_p$, the sets $U_{\mathbb{F}}(n) := 1 + \mathfrak{p}^n$ is open and closed (and compact as it is bounded). Indeed it can be described as

$$U_{\mathbb{F}}(n) = \{x : |x-1|_p \leq p^{-n}\} = \{x : |x-1|_p < p^{-n+1}\}.$$

For $n = 0$, we define by convention that $U_{\mathbb{F}}(0) = O^\times$.

There are two commonly used Haar measures:

(1) On $\mathbb{Q}_p$ we have the additive Haar measure dx so that for any compact open subset S and $y \in \mathbb{F}$, $\text{Vol}(S, dx) = \text{Vol}(y + S, dx)$. We normalize it so that $\text{Vol}(O, d\mu) = 1$.
(2) On $\mathbb{Q}_p^\times$ we have the multiplicative Haar measure $d^\times\mu$ so that $\text{Vol}(S, d^\times x) = \text{Vol}(yS, d^\times x)$ for any $y \in \mathbb{F}^\times$. We normalize it so that $\text{Vol}(O^\times, d^\times\mu) = 1$.

2.2. **Characters of p-adic fields.**

Definition 2.1. The additive character $\psi_p : \mathbb{Q}_p \to \mathbb{C}^\times$ is a continuous function such that $\psi_p(x+y) = \psi_p(x)\psi_p(y)$.

Lemma 2.2. *There exists c such that ψ_p is constant 1 on $p^c\mathbb{Z}_p$.*

Let $c = c(\psi_p)$ be minimal integer with the property that $\psi_p(p^c\mathbb{Z}_p) = 1$. We shall fix ψ_p so that $c(\psi_p) = 0$, in which case we call ψ_p unramified.

Example 2.3. For $x = \sum_{i \geq i_0} a_i p^i$, define $x_{<0} = \sum_{i_0 \leq i < 0} a_i p^i$ which can be 0 if $i_0 \geq 0$. Then the function

$$\psi_p(x) = e^{2\pi i x_{<0}}$$

is an additive character with $c(\psi_p) = 0$.

Lemma 2.4. *Any additive character of $\mathbb{Q}_p$ is of form $\psi_a(x) = \psi_p(ax)$ for some $a \in \mathbb{Q}_p$. In other words, the Pontryagin dual of $\mathbb{Q}_p$ is $\mathbb{Q}_p$ itself (we write $\widehat{\mathbb{Q}_p} = \mathbb{Q}_p$ for this).*

Proof. The basic strategy is to reduce to the Pontryagin duality of finite groups. We skip the details here. □

Remark 2.5. For any field extension $\mathbb{E}/\mathbb{Q}_p$, one can construct an additive character on it by $\psi_p \circ \text{Tr}_{\mathbb{E}/\mathbb{Q}_p}$, and obtain similar Pontryagin duality.

Definition 2.6. A multiplicative character $\chi_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ is a continuous function such that $\chi_p(xy) = \chi_p(x)\chi_p(y)$. Again χ_p must be locally constant. $c = c(\chi_p) \geq 0$ is defined to be the smallest integer such that $\chi_p|_{U_{\mathbb{F}}(c)}$ is constant 1. We call χ_p unramified if $c(\chi_p) = 0$, and ramified otherwise. The conductor of χ_p is $C(\chi_p) = p^{c(\chi_p)}$.

2.3. **Lie algebra description.** In a small enough neighborhood, one can associate additive character and multiplicative character as follows: Suppose p is large enough, then we have p-adic exponential map:

$$\begin{aligned} \exp : \mathfrak{p} &\to U_{\mathbb{F}}(1) \\ x &\mapsto \exp(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots \end{aligned}$$

This is convergent for p large enough and is actually a bijection with inverse map being p-adic logarithm

$$\log : U_{\mathbb{F}}(1) \to \mathfrak{p}$$
$$1 + x \mapsto \log(1 + x) = x - \frac{x^2}{2} + \cdots$$

We consider the domain of exp as a neighborhood in the Lie algebra of $\mathbb{F}$ and $U_{\mathbb{F}}(1)$ as a neighborhood in the Lie group $\mathbb{F}^{\times}$.

As the exponential map change additive structure to multiplicative structure, we have the following

Corollary 2.7. *Let p be large enough and χ be a multiplicative character of $\mathbb{Q}_p^{\times}$ with $c = c(\chi_p)$. Let ψ be an unramified additive character. Then there exists $\alpha_\chi \in O^{\times}$ such that*

$$\chi \circ \exp(x) = \psi(p^{-c}\alpha_\chi x)$$

for any $x \in \mathfrak{p}$. In particular if $v_p(x) \geq c/2$, we have

$$\chi(1 + x) = \psi(p^{-c}\alpha_\chi x).$$

2.4. **Adele.**

Definition 2.8. Let

$$\begin{aligned}\mathbb{A} &= \{(x_v)_v | x_v \in \mathbb{R} \text{ if } v = \infty, x_v \in \mathbb{Q}_p \text{ if } v = p, \text{ and } x_p \in \mathbb{Z}_p \text{ for a.a. (almost all) } p\} \\ &= \prod{}' \mathbb{Q}_v\end{aligned}$$

be the ring of adeles, with component-wise addition and multiplication. Its open subsets are constructed from the following standard ones:

$$U = \prod{}' U_v, \text{ where } U_v \subset \mathbb{Q}_v \text{ are open subsets, and } U_p = \mathbb{Z}_p \text{ for a.a. } p.$$

Lemma 2.9 (Approximation)**.** *$\mathbb{A} = \mathbb{Q} + U$ where $U = [0, 1) \times \prod_p \mathbb{Z}_p$*

Proof. By Chinese remainder theorem, for any $x \in \mathbb{A}$ there exists $q \in \mathbb{Q}$ such that $(x - q)_p \in \mathbb{Z}_p$. One can then further adjust q by an integer so that $(x - q)_\infty \in [0, 1)$. Lastly notice that $\mathbb{Q} \cap U = \{0\}$ so the decomposition is unique. □

Corollary 2.10. *If we give $\mathbb{A}$ the product measure using the additive Haar measure for each place $\mathbb{Q}_v$, and the quotient measure for $\mathbb{Q}\backslash\mathbb{A}$, then Vol$(\mathbb{Q}\backslash\mathbb{A}) = 1$.*

Definition 2.11. An additive character ψ on $\mathbb{Q}\backslash\mathbb{A}$ is a character $\psi = \otimes\psi_v : \mathbb{A} \to \mathbb{C}^{\times}$ such that ψ_v are continuous character of $\mathbb{Q}_v$ and $\psi(a + x) = \psi(x)$ for any $a \in \mathbb{Q}$.

Such character exists. For examle one can pick for any p-adic place p, $\psi_p(x) = e^{2\pi i x_{<0}}$, and at ∞, $\psi_\infty(x) = e^{-2\pi i x}$. In the following we denote by ψ the resulting additive character of $\mathbb{Q}\backslash\mathbb{A}$.

As for local version, we have

Lemma 2.12 (Pontryagin duality)**.** *Any additive character of $\mathbb{Q}\backslash\mathbb{A}$ is of form $\psi_a(x) = \psi(ax)$. That is $\widehat{\mathbb{Q}\backslash\mathbb{A}} = \mathbb{Q}$.*

Proof. On one hand given ψ as above, we need to check that $\psi(a) = 1$ for all $a \in \mathbb{Q}$. Then ψ_a would indeed be a character of $\mathbb{Q}\backslash\mathbb{A}$. Any $a \in \mathbb{Q}$ can be written as

$$a = n + \sum_p \sum_{i_p \leq i < 0} a_{p,i} p^i$$

where n is the integer and $0 \leq a_{p,i} < p$. Then $\psi_l(a) = e(2\pi i \sum_{i_l \leq i < 0} a_{l,i} p^i)$ for any finite place l, $\psi_\infty(a) = e(-2\pi i \sum_p \sum_{i_p \leq i < 0} a_{p,i} p^i)$. One can easily check that

$$\prod_v \psi_v(a) = 1.$$

□

We skip the proof of the other direction here. Using this we can do Fourier analysis on $\mathbb{Q}\backslash\mathbb{A}$ as follows.

Theorem 2.13. *For any $f \in C(\mathbb{Q}\backslash\mathbb{A})$, we have*

$$f = \sum_{a \in \mathbb{Q}} c_a \psi_a$$

where $c_a = \int_{x \in \mathbb{Q}\backslash\mathbb{A}} f(x)\psi(-ax)dx$.

Proof. It follows from general Fourier analysis when Pontryagin duality is known. It can also be derived from the Fourier analysis for $C[0, 1]$ using approximation result. □

2.5. **Idele.**

Definition 2.14. The group of ideles is

$$\mathbb{A}^\times = \prod{}' \mathbb{Q}_v^\times = \{(x_v) | x_v \in \mathbb{Q}_v, x_p \in \mathbb{Z}_p^\times \text{ for a.a. } p\}.$$

The basis for its topology is $V = \prod{}' V_v$ where $V_v = \mathbb{Z}_p^\times$ for a.a. p. We give product measure for $\mathbb{A}^\times$ using multiplicative Haar measure of $\mathbb{Q}_v^\times$, and quotient measure for $\mathbb{Q}^\times\backslash\mathbb{A}^\times$.

Lemma 2.15 (Approximation for ideles)**.** *We have*

$$\mathbb{A}^\times = \mathbb{Q}^\times(\mathbb{R}_+ \prod \mathbb{Z}_p^\times).$$

Here we can not use smaller region at ∞ compared to adeles.

A multiplicative character $\chi : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ is necessarily a product of local characters

$$\chi = \prod_v \chi_v$$

which is further invariant by $\mathbb{Q}^\times$. They can be viewed as automorphic form for GL_1, and are generalizations of Dirichlet characters.

3. Basics for groups

Let M_n be the ring of rank n square matrices, and let $G = \mathrm{GL}_n$ be the general linear group defined over $\mathbb{Q}$, and $G(\mathbb{Q}_v)$ be its $\mathbb{Q}_v$-points. For each v we fix a maximal compact subgroup K_v: $K_\infty = O(n)$ and $K_p = \mathrm{GL}_n(\mathcal{O})$ (determinant must be in $\mathcal{O}^\times$).

Let Z be the center of G and $\mathrm{PG} = \mathrm{PGL}_n = Z\backslash G$.

For each partition $n = n_1 + n_2 + \cdots n_k$ and $\underline{n} = (n_1, \cdots, n_k)$, we denote by $P_{\underline{n}}$ the group of blockwisely upper triangular matrices whose diagonal blocks are rank n_i square matrices, with its Levi subgroup $L_{\underline{n}}$ consisting of only diagonal blocks. In the special case $n_i = 1$, we recover the Borel subgroup B and diagonal matrices A. Let N denote the corresponding unipotent subgroup for B.

Here are some basic decomposition results for matrices in G, though the exact meanings differ for archimedean place ∞ and p-adic places.

Lemma 3.1. *We skip the local field from our notations, but everything is understood locally.*

1. *Iwasawa decomposition* $G = BK$.
2. *Bruhat decomposition* $G = \coprod_{w \in W} BwN$, *where* W *is the Weyl group (group of permutation matrices for* GL_n*).*
3. *Cartan decomposition* $G = KAK$.

3.1. Measure.

$G(\mathbb{Q}_v)$ has Haar measure $d\mu$ so that we can do change of variable for integrals

$$\int_G f(xg) d\mu(g) = \int_G f(gx) d\mu(g) = \int_G f(g) d\mu(g).$$

Indeed one can check that

$$d\mu(g) = \frac{d\mu_{M_n}(g)}{\det^n(g)}$$

is the Haar measure where $d\mu_{M_n}$ is the additive Haar measure on M_n.

On the other hand, $P_{\underline{n}}$ and B do not have Haar measures. They have left/right Haar measures which allow change of variable on left or right, but these measures are not consistent with each other.

Definition 3.2. The modular character Δ_G of a group G is a multiplicative character on G such that

$$d\mu_L(gx) = d\mu_L(g)\Delta_G(x).$$

Example 3.3. Consider $B \subset \mathrm{GL}_2$, where any element takes the form $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. Then the left Haar measure on B is given by

$$d\mu_L(g) = d^\times a \, d^\times d \, \frac{db}{|a|_v}.$$

Indeed with a change of variable by $g' = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$, we get that

$$\begin{aligned} d\mu_L(g'g) = d\mu_L\left(\begin{pmatrix} ax & bx+dy \\ 0 & zd \end{pmatrix}\right) &= d^\times(ax)d^\times(zd)\frac{d(bx+dy)}{|ax|_v} \\ &= d^\times a d^\times d\frac{d(bx)}{|ax|_v} \\ &= d^\times a d^\times d\frac{db}{|a|_v}. \end{aligned}$$

To obtain the modular character, we make a change of variable on the right by g' and compute that

$$\begin{aligned} d\mu_L(gg') = d\mu_L\left(\begin{pmatrix} ax & ay+bz \\ 0 & zd \end{pmatrix}\right) &= d^\times(ax)d^\times(zd)\frac{d(ay+bz)}{|ax|_v} \\ &= d^\times a d^\times d\frac{d(bz)}{|ax|_v} \\ &= d^\times a d^\times d\frac{db}{|a|_v}\frac{|z|_v}{|x|_v}. \end{aligned}$$

Thus the modular character is

$$\Delta_B(g') = \frac{|z|_v}{|x|_v}$$

for GL_2.

More generally we have the following:

Lemma 3.4. *Let*

$$g = \begin{pmatrix} M_{11} & N_{12} & \cdots & N_{1k} \\ 0 & M_{22} & \cdots & N_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & M_{kk} \end{pmatrix} \in P_{\underline{n}},$$

then

$$\Delta_{P_{\underline{n}}}(g) = \prod_{1\le i\le k} |\det(M_{ii})|_v^{\sum_{j<i} n_j - \sum_{j>i} n_j}.$$

3.2. **Adelic group and approximation.**

Definition 3.5. Let

$$G(\mathbb{A}) = \prod{}' G(\mathbb{Q}_v)$$

where $g = (g_v) \in G(\mathbb{A})$ satisfies $g_p \in K_p$ for a.a. p.

Denote

$$[G] = G(\mathbb{Q})\backslash G(\mathbb{A}).$$

The following approximation result for is slightly more complicated, and can be found in [2] for GL_2.

Lemma 3.6. *Let K_0 be an open compact subgroup of $G(\mathbb{A}_{fin}) = \prod_{p\text{-adic places}}{}' G(\mathbb{Q}_p)$, such that the set* $\det(K_0) = \prod_p \mathbb{Z}_p^\times$. *Then*

$$G(\mathbb{A}) = G(\mathbb{Q})(G_+(\mathbb{R}) \times K_0).$$

Here $G_+(\mathbb{R})$ is the subgroup of $G(\mathbb{R})$ with positive determinant.

Proof. This follows from known strong approximation for semi-simple groups SL_n, and approximation for $\mathbb{A}^\times$. □

Corollary 3.7. *Similar approximation result holds for $PG = \mathrm{PGL}_2$ and we have the following identification*

$$PG(\mathbb{Q})\backslash PG(\mathbb{A})/(\prod PG(\mathbb{Z}_p)) \to SL_2(\mathbb{Z})\backslash PG_+(\mathbb{R})$$
$$g = \gamma g_\infty g_{fin} \mapsto g_\infty$$

Here $\gamma \in PG(\mathbb{Q})$.

Proof. It is clearly surjective. We check it's well-defined and injective. We claim first that

$$\mathrm{PG}(\mathbb{Q}) \cap (\mathrm{PG}_+(\mathbb{R}) \times \prod \mathrm{PG}(\mathbb{Z}_p)) = \mathrm{SL}_2(\mathbb{Z})$$

This is because for any $\gamma \in \mathrm{PG}(\mathbb{Q})$, $\gamma_p \in \mathrm{PG}(\mathbb{Z}_p)$ implies that, there exists $\alpha \in Z(\mathbb{Q})$ such that $\alpha\gamma \in G(\mathbb{Z}_p)$ for all p. This means that $\alpha\gamma \in \mathrm{GL}_2(\mathbb{Z})$ with $\det(\alpha\gamma) = \pm 1$. The requirement $\alpha\gamma \in \mathrm{PG}_+(\mathbb{R})$ further implies that $\det(\alpha\gamma) = 1$. Thus the claim follows.

Now if

$$g = \gamma_1 g_{1,\infty} g_{1,fin} = \gamma_2 g_{2,\infty} g_{2,fin}$$

are two decompositions, then

$$\gamma_2^{-1}\gamma_1 = g_{2,\infty} g_{1,\infty}^{-1} g_{2,fin} g_{1,fin}^{-1} \in \mathrm{PG}(\mathbb{Q}) \cap (\mathrm{PG}_+(\mathbb{R}) \times \prod \mathrm{PG}(\mathbb{Z}_p)) = \mathrm{SL}_2(\mathbb{Z}).$$

This confirms that the map in the corollary is well-defined and injective. □

We shall give $G(\mathbb{A})$ the product measure from Haar measure of $G(\mathbb{Q}_v)$ as before and $[G] := G(\mathbb{Q})\backslash G(\mathbb{A})$ the quotient measure. Note that $[G]$ does not have finite volume but [PG] has.

3.3. **Unitary groups.** Let $\mathbb{K}$ be a quadratic field extension of $\mathbb{Q}$, and $\tau \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ be a non-trivial involution. Let $(V, <,>)$ be a Hermitian space over $\mathbb{K}$ with Hermitian form $<,>$, which is linear/$\tau-$linear in first/second variable. In this setting let

$$G = U(V, <,>) = \{g \in \mathrm{GL}(V), < gx, gy >=< x, y > \ \forall x, y \in V\}.$$

Many discussions for GL_n also hold for $U(V)$. Actually these two groups are isomorphic at split places as explained below. From algebraic number theory, there are infinite number of places where $\mathbb{K}_p \simeq \mathbb{Q}_p \times \mathbb{Q}_p$.

Lemma 3.8. *Suppose that $\mathbb{K}_p \simeq \mathbb{Q}_p \times \mathbb{Q}_p$. Then $G(\mathbb{Q}_p) \simeq GL_n(\mathbb{Q}_p)$.*

Proof. We use the identification $\mathbb{K}_p = \mathbb{Q}_p \times \mathbb{Q}_p$. Let $\tau_p \in \mathrm{Gal}(\mathbb{K}_p/\mathbb{Q}_p)$ be the map swapping two coordinates of $\mathbb{K}_p$ which coincides with τ when restricted to $\mathbb{K}$. We fix an ONB(orthonormal basis) for V, define $V_p = V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and extend $<,>$ to V_p. Then

$$\begin{aligned} G(\mathbb{Q}_p) &= \{g \in \mathrm{GL}(V_p) | < gx, gy >=< x, y > \ \forall x, y \in V_p\} \\ &= \{g | g\tau_p(g^T) = I\} = \{(g_1, g_2) | (g_1, g_2)(g_2^T, g_1^T) = I\} \\ &= \{(g_1, (g_1^T)^{-1})\} \simeq \mathrm{GL}_n(\mathbb{Q}_p). \end{aligned}$$

Here in second line we use the ONB for V as ONB for V_p. □

Remark 3.9. Later on we shall make use the following features of unitary groups:

(1) $[G]$ not only has finite volume but is also compact.
(2) At places of interest to us, we can however make assumptions to reduce to classification of representations of GL_n, which might be better understood.

4. Automorphic forms and representations

4.1. **Starting from classical modular forms.** Let $\mathbb{H}$ be the upper half plane with hyperbolic measure $\frac{dxdy}{y^2}$ and Laplacian

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

$\mathrm{PGL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by

$$g.z = \frac{ax+b}{cz+d}$$

if g is represented by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let

$$\Gamma_0(N) = \{g \in \mathrm{SL}_2(\mathbb{Z}) | g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } c \equiv 0 \mod (N)\}$$

and $\Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z})$. For simplicity we focus on Maass forms, though holomorphic modular forms are also similar.

Definition 4.1. A Maass form of Laplace eigenvalue λ and level N is a function $f : \mathbb{H} \to \mathbb{C}$ such that

(1) $\Delta f = \lambda f$;
(2) $f(\gamma.z) = f(z)\ \forall \gamma \in \Gamma_0(N)$;
(3) f has polynomial growth at cusps (Think of cusps as directions you can go to infinity for $\Gamma_0(N)\backslash\mathbb{H}$; for example, we can let $y \to \infty$ and require $f(x+iy) \ll y^A$ for some fixed A. Though there are other directions for general N.).

For such modular forms we have

(1) Petersson inner product
$$< f_1, f_2 >= \frac{1}{[\Gamma_0(N) : \mathrm{SL}_2(\mathbb{Z})]} \int_{\Gamma_0(N)\backslash\mathbb{H}} f_1(z)\overline{f_2(z)}\frac{dxdy}{y^2}.$$
(2) Fourier expansion. As $f(z+1) = f(z)$, we can write
$$f(z) = \sum_{n\in\mathbb{Z}} \lambda_f(n) y^{1/2} K_\lambda(2\pi|n|y)e^{2\pi inx}.$$
Here $\lambda_f(n)$ is the $n-$th Fourier coefficient of f and K_λ is some $K-$Bessel function with parameter related to Laplace eigenvalue λ.
(3) Hecke operator for $p \nmid N$
$$T_p f(z) = \sum_{\gamma\in\Gamma_0(N)\backslash\Gamma_0(N)\begin{pmatrix} p & \\ & 1 \end{pmatrix}\Gamma_0(N)} f(\gamma.z).$$
(4) L-function which is first defined for $\mathrm{Re}(s)$ large enough:
$$L(s, f) = \sum_{n\geq 0} \lambda_f(n) n^{-s}$$
which admits analytic continuation and functional equation.

4.2. **Changing to automorphic forms.** We first define local compact subgroups

$$K_0(p^i) = \{g \in K_p | g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } c \equiv 0 \mod p^i\}.$$

Globally if $N = \prod p^{a_p}$, define

$$K_0(N) = \prod_{p|N} K_0(p^{a_p}) \cdot \prod_{p\nmid N} K_p,$$

which is a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_{fin})$. By abuse of notation we also denote by $K_0(N)$ its image in $\mathrm{PGL}_2(\mathbb{A}_{fin})$, which again is a compact open subgroup.

Going along Corollary 3.7, we get

Lemma 4.2. *We have the identification*

$$PG(\mathbb{Q})\backslash PG(\mathbb{A})/(\mathrm{SO}(2) \times K_0(N)) \to \Gamma_0(N)\backslash PG_+(\mathbb{R})/\mathrm{SO}(2) \to \Gamma_0(N)\backslash\mathbb{H}$$
$$g = \gamma g_\infty g_{fin} \mapsto g_\infty \mapsto g_\infty.i$$

Proof. Here we replaced $\prod \mathrm{PG}(\mathbb{Z}_p)$ by $K_0(N)$, with the observation that same proof as before implies

$$\mathrm{PG}(\mathbb{Q}) \cap (\mathrm{PG}_+(\mathbb{R}) \times K_0(N)) = \Gamma_0(N).$$

It remains to see that $g.i = i$ for $g \in \mathrm{PGL}_{2,+}(\mathbb{R})$ iff $g \in \mathrm{SO}(2)$ up to a scalar. □

Now with a Maass form f viewed as a function on $\Gamma_0(N)\backslash\mathbb{H}$, we can associated a function F on $[\mathrm{PGL}_2]$ by

$$F(g) = f(g_\infty.i), \text{ if } g = \gamma g_\infty g_{fin}.$$

F will be automatically left invariant by $\mathrm{PG}(\mathbb{Q})$ and right invariant by $\mathrm{SO}(2) \times K_0(N)$. Conversely given F on $[\mathrm{PG}]$ which is further invariant by $\mathrm{SO}(2) \times K_0(N)$, we can recover a modular form by

$$f(x + iy) = F(g_\infty)$$

where $g_\infty = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$.

The automorphic variant of topics for modular forms are as follows:

(1) Petersson inner product can be given now by
$$< F_1, F_2 >= \int_{[\mathrm{PGL}_2]} F_1(g)\overline{F_2(g)}dg.$$
(2) Fourier expansion becomes Whittaker expansion, to be discussed later.
(3) Hecke operators becomes Hecke algebra action as follows: Let $\mathbb{F}$ be either $\mathbb{A}$ or any local field $\mathbb{Q}_v$. For a test function $f \in S(G(\mathbb{F}))$, the space of Schwartz function (smooth and rapidally decay; in the case of p-adic field, the latter actually requires compact support), we can define an operator on the space of automorphic forms by
$$\rho(f)F(x) = \int_{x\in G(\mathbb{F})} f(g)F(xg)dg.$$
$S(G(\mathbb{A}))$ becomes an algebra under usual addition and convolution, and it's straightforward to check that $\rho(f_1 * f_2) = \rho(f_1) \circ \rho(f_2)$. In particular if $\mathbb{F} = \mathbb{Q}_p$ and f is the characteristic function of double coset $K_p \begin{pmatrix} p & \\ & 1 \end{pmatrix} K_p$, one recovers the Hecke operator up to a scalar.

(4) L-functions can be defined as before using Fourier/Whittaker expansions, or can be related to the so-called period integrals, similar to Tate's thesis. We will also give some more complicated examples later on.

In general we consider higher rank group G and we don't require F to be invariant by such a compact subgroup, but just require it to be $K-$finite for $K = \prod K_v$. Here $K-$finite means that when we let K acts on F by right translation, we get a finite dimensional vector space.

We view the following (very vague) definition of automorphic forms as generalization of modular forms

Definition 4.3. An automorphic form (with trivial central character) is a smooth function $F : [G] \to \mathbb{C}$ which is $K-$finite and has moderate growth at cusps. Denote the linear space of such automorphic forms by $\mathcal{A}([G])$

4.3. **Automorphic representation.** We start with some general notations of representations. Given a group G, by a (complex) representation of G, we mean a group homomorphism

$$\pi : G \to \mathrm{GL}(V)$$

for some (complex) vector space V.

A subrepresentation of (π, V) is a subspace $W \subset V$ which is invariant under any $\pi(g)$.

π is called irreducible if it has no non-trivial subrepresentations. In that case, for any non-zero $v \in V$, $\mathrm{span}\{\pi(g)v\} = V$.

Given now an automorphic form $F \in \mathcal{A}([G])$, and $g \in G(\mathbb{A})$, we obtain a new automorphic form from right translation $g.F(x) = F(xg)$. The representation generated by F: $\pi = \mathrm{span}\{g.F\}$ becomes a representation of $G(\mathbb{A})$.

Such representations can be very complicated in general. In favorable cases we hope that there exists π_v local representations of each $G(\mathbb{Q}_v)$ so that $\pi \simeq \otimes\pi_v$. To this end we need some more notations.

Definition 4.4. Consider a p-adic place $\mathbb{Q}_p$ and π a representation of $G(\mathbb{Q}_p)$. π is called admissible if for any compact open subgroup K', the dimension of $K'-$invariant vectors $\dim(\pi^{K'}) < \infty$.

At $\mathbb{R}$ we usually consider (π, V) to be so-called $(\mathfrak{g}, K)-$module, where

(1) $\mathfrak{g}$ is the Lie algebra of $G(\mathbb{R})$ acting on V. It does not require whole $G(\mathbb{R})$ to act on V actually, but when it does have $G(\mathbb{R})$ action, then $x \in \mathfrak{g}$ acts on $v \in V$ by

$$x.v = \lim_{t\to 0} \frac{\pi(e^{xt})v - v}{t}.$$

One can roughly think of this as smoothness.

(2) $\pi|_K$ is admissible in the sense that the restriction of π to K

$$\pi|_K = \bigoplus_{k\in\mathbb{Z}} V_k$$

where weight k subspace V_k is finite dimensional and SO(2) acts on $v \in V_k$ by

$$\pi\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right) v = e^{ik\theta} v.$$

(3) Two actions above are consistent in the sense that

$$\pi(k)(x.(\pi(k^{-1})v)) = (\mathrm{Ad}_k x).v,$$

where $\mathrm{Ad}_k x = kxk^{-1}$ is an element in $\mathfrak{g}$.

Globally let π be a representation of $G(\mathbb{A})$

(1) π is called smooth is so is $\pi|_{G(\mathbb{Q}_p)}$ and it is a $\mathfrak{g}-$module;
(2) π is called admissible if for any compact subgroup of form $K' = \mathrm{SO}(2) \times K'_{fin}$, the weight k and $K'_{fin}-$invariant subspace of π is finite dimensional.

Remark 4.5. These conditions follow directly from automorphic forms obtained from Maass forms, and are thus natural to impose.

Definition 4.6. Given π_v irreducible (smooth) admissible representations of $G(\mathbb{Q}_v)$ for all v, such that for a.a. p, $\dim(\pi_p^{K_p}) = 1$ spanned by some fixed element φ_p°, then the restricted tensor product of π_v is

$$\prod{}' \pi_v = \left\{ (\varphi_v) \in \prod \pi_v | \varphi_v = \varphi_0^\circ \right\}.$$

Theorem 4.7 (Bump 3.3.3)**.** *If* (π, V) *is an irreducible (smooth) admissible representation of* $G(\mathbb{A})$*, then there exists irreducible* $(\mathfrak{g}, K)-$*module* π_∞ *and (smooth) admissible representation* π_p *such that*

$$\pi \simeq \prod{}' \pi_v.$$

This result partially reduces the study of global representation to that of local representation, as it is possible to give full classification of local representations.

On the other hand, given all local representations, the resulting restricted tensor product is **not** necessarily an automorphic representation, the main issue being that it may not have $G(\mathbb{Q})-$invariance. To achieve that one basically need all relevant L-functions have analytic continuation and functional equation.

4.4. **Matrix coefficient.** Note that we have $\pi \simeq \prod{}' \pi_v$, not directly equal. In other words, we can not pick elements $\varphi_v \in \pi_v$ and say that global automorphic form satsifes $\varphi = \prod \varphi_v$.

There are however two other objects for which we can say the global version equals the product of local versions (up to a constant).

We consider for an irreducible admissible representation π realized on vector space V. The $\hat{V}$ be the smooth dual for V and $\hat{\pi}$ be the contragredient representation of G on $\hat{V}$ such that for the natural pairing $<,>$ between V and $\hat{V}$, we have

$$< \pi(g)v, \hat{\pi}(g)\hat{v} >=< v, \hat{v} >$$

for any $v \in V, \hat{v} \in \hat{V}, g \in G$.

We have the following multiplicity one result

Lemma 4.8. *Let* $\mathbb{F}$ *be* $\mathbb{Q}_v$ *or* $\mathbb{A}$*,* π *be an irreducible (smooth) admissible representation of* $G(\mathbb{F})$*, then* $\mathrm{Hom}_{G(\mathbb{F})}(\pi \times \hat{\pi}, \mathbb{C})$ *is* $1-$*dimensional.*

Proof. Consider first $\mathbb{F}$ is a local field. We have in general

$$\mathrm{Hom}_{G(\mathbb{F})}(\pi \times \hat{\pi}, \mathbb{C}) = \mathrm{Hom}_{G(\mathbb{F})}(\pi, \pi)$$

which is 1-dimensional by Schur's lemma. For the global field, note that

$$\mathrm{Hom}_{G(\mathbb{A})}(\pi \times \hat{\pi}, \mathbb{C}) = \prod \mathrm{Hom}_{G(\mathbb{Q}_v)}(\pi_v \times \hat{\pi}_v, \mathbb{C}),$$

which is also 1-dimensional. □

Corollary 4.9. *Fix nontrivial pairings $<,>$ between $\pi, \hat{\pi}$ and $<,>_v$ between $\pi_v, \hat{\pi}_v$ for any v. Then there exists a nonzero multiple c so that for any $\varphi = \otimes\varphi_v \in \pi$ and $\phi = \otimes\phi_v \in \hat{\pi}$, we have*

$$< \varphi, \phi >= c \prod < \varphi_v, \phi_v >_v .$$

Remark 4.10. Note one can adjust normalization for the pairings so that $c = 1$.

Definition 4.11. Given $(\pi, V), (\hat{\pi}, \hat{V}), \varphi \in V, \phi \in \hat{V}$, we define the associated matrix coefficient

$$\Phi_{\varphi,\phi}(g) =< \pi(g)\varphi, \phi > .$$

Corollary 4.12. *Let $\Phi_{\varphi,\phi}(g)$ be the matrix coefficient for global automorphic form φ, ϕ, and Φ_{φ_v,ϕ_v} be the corresponding local matrix coefficient, then there exists constant c such that*

$$\Phi_{\varphi,\phi}(g) = c \prod_v \Phi_{\varphi_v,\phi_v}(g_v).$$

Remark 4.13. When π is further unitary with pairing $(,)_\pi$, as in the case with Petersson inner product, by Riesz representation theory there exists $\varphi' \in \pi$ such that

$$(\varphi, \varphi')_\pi =< \varphi, \phi >,$$

then one can identify the matrix coefficient as

$$\Phi_{\varphi,\varphi'}(g) = (\pi(g)\varphi, \varphi')_\pi.$$

One can similarly write global matrix coefficient as product of local matrix coefficients. We shall be mainly interested in the case $\varphi' = \varphi$.

4.5. **Whittaker/Fourier expansion.** We consider $G = \mathrm{GL}_2$ case first. Then we have

$$\varphi(g) = \varphi\left(\begin{pmatrix}1 & n\\ & 1\end{pmatrix} g\right)|_{n=0} =: \phi(n)|_{n=0}$$

where we consider ϕ as a function on $\mathbb{A}$ which is clearly invariant by $\mathbb{Q}$. Applying Theorem 2.13 for ϕ, we get that

$$\phi(n) = \sum_{a\in\mathbb{Q}} \hat{\phi}(a)\psi_a(n), \varphi(g) = \sum_{a\in\mathbb{Q}} \hat{\phi}(a)$$

where

$$\begin{aligned}\hat{\phi}(a) &= \int_{n\in\mathbb{Q}\backslash\mathbb{A}} \phi(n)\psi_a(-n)dn = \int_n \varphi\left(\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} g\right)\psi(-an)dn\\ &= \int_n \varphi\left(\begin{pmatrix}1 & a^{-1}n\\ 0 & 1\end{pmatrix} g\right)\psi(-n)dn = \int_n \varphi\left(\begin{pmatrix}a^{-1} & \\ & 1\end{pmatrix}\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & \\ & 1\end{pmatrix} g\right)\psi(-n)dn\\ &= \int_n \varphi\left(\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & \\ & 1\end{pmatrix} g\right)\psi(-n)dn.\end{aligned}$$

In the last line we used that φ is left-invariant by rational matrices.

Definition 4.14. The (global) Whittaker function associated to φ is defined to be

$$W_\varphi(g) = \int_{n\in\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix}1 & n\\ & 1\end{pmatrix} g\right)\psi(-n)dn.$$

Corollary 4.15. *We have Whittaker expansion*

$$\varphi(g) = \int_{n\in\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix}1 & n\\ & 1\end{pmatrix}g\right)dn + \sum_{a\in\mathbb{Q}^\times} W_\varphi\left(\begin{pmatrix}a & \\ & 1\end{pmatrix}g\right).$$

The first term corresponds to the 0−th Fourier coefficient, which motivates the following general definition:

Definition 4.16. φ is called a cusp form if

$$\int_{n\in\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix}1 & n\\ & 1\end{pmatrix}g\right)dn = 0$$

for any g.

To understand the remaining terms, note that by a change of variable, we have

$$W_\varphi\left(\begin{pmatrix}1 & m\\ & 1\end{pmatrix}g\right) = \psi(m)W_\varphi(g).$$

The right translation of G is also preserved when we compute W_φ from φ. This motivates the following definition

Definition 4.17. Given a representation π of $G(\mathbb{F})$, the global/local Whittaker functional is a $G(\mathbb{F})-$equivalent map

$$\begin{aligned}\mathcal{W}: \pi &\to C^\infty(N(\mathbb{F})\backslash G(\mathbb{F}), \psi)\\ \varphi &\mapsto W_\varphi\end{aligned}$$

Here $C^\infty(N(\mathbb{F})\backslash G(\mathbb{F}), \psi)$ is the space of smooth functions f on $G(\mathbb{F})$ satisfying

$$f(\begin{pmatrix}1 & m\\ & 1\end{pmatrix}g) = \psi(m)f(g).$$

Lemma 4.18. *For $\mathbb{F}$ a local field and π an irreducible (smooth) admissible representation of $G(\mathbb{F})$,*

$$\dim \mathrm{Hom}_{G(\mathbb{F})}(\pi, C^\infty(N(\mathbb{F})\backslash G(\mathbb{F}), \psi)) \leq 1.$$

This can be achieved once full classification of local irreducible (smooth) admissible representation is done. We shall skip the details here.

Corollary 4.19. *Suppose $\pi \simeq \otimes'\pi_v$. Then $\dim \mathrm{Hom}_{G(\mathbb{A})}(\pi, C^\infty(N(\mathbb{A})\backslash G(\mathbb{A}), \psi)) \leq 1$, with equality iff $\dim \mathrm{Hom}_{G(\mathbb{Q}_v)}(\pi_v, C^\infty(N(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v), \psi)) = 1$ for all v. In that case, there exists a constant c such that for any $\varphi \simeq \prod \varphi_v \in \pi$,*

$$W_\varphi(g) = c\prod W_{\varphi,v}(g_v).$$

Here $W_{\varphi,v}$ is the image of φ_v under $\mathcal{W}_v \in \mathrm{Hom}_{G(\mathbb{Q}_v)}(\pi_v, C^\infty(N(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v), \psi))$.

Now for general GL_n, let N_n denote the subgroup consisting of upper triangular unipotent matrices. Let ψ_0 be an additive character on $u = (u_{ij}) \in N_n$ given by

$$\psi_0(u) = \psi\left(\sum_{2\leq i\leq n} u_{i-1,i}\right).$$

From, for example, [5, Lecture 4], we have

Lemma 4.20. *For $G = GL_n$, and $\varphi \in \mathcal{A}([G])$ a cuspidal automorphic form, we have*

$$\varphi(g) = \sum_{\gamma \in N_{n-1}(\mathbb{Q}) \backslash GL_{n-1}(\mathbb{Q})} W_\varphi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right),$$

where

$$W_\varphi(g) = \int_{u \in [N_n]} \varphi(ug)\psi_0(-u) du.$$

Note that taking $n = 2$ we recover the formula for GL_2.

5. Local representations

Here we give a brief introduction to the classification of local irreducible smooth admissible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$, and give some explicit computations to be used later on.

For the parabolic induction part, the main reference is [1]; For supercuspidal representation part, one can read [3] for GL_2 case, and [4] for GL_n case.

5.1. Induction of representation.

Given a subgroup $H < G$, a representation $\lambda : H \to \mathrm{GL}(V)$, an important way to construct a representation is by induction

$$\pi = \mathrm{Ind}_H^G \lambda = \{f : G \to V | f(hg) = \lambda(h) f(g),\ f \text{ smooth}\}.$$

Here we impose additional requirement of f smooth as we only care about smooth representations. It may however be not irreducible/admissible, so one need to choose (H, λ) carefully. Our goal is twofold: describe the construction, and describe the corresponding conductor.

Everything here is local so we skip subscript v or p.

5.2. Parabolic induction.

Recall for the partition $n = \sum n_i$, the corresponding Levi subgroup is $L_{\underline{n}} \simeq \mathrm{GL}_{n_1} \times \cdots \mathrm{GL}_{n_k}$. Let λ_i be irreducible smooth admissible ('Good' in short) representation of GL_{n_i} realized on vector space V_i. Then $\lambda := \otimes \lambda_i$ is a Good representation of $L_{\underline{n}}$ realized on $V = \otimes V_i$. It can further be extended to a representation of $P_{\underline{n}}$ on the same vector space, such that for any $p = mn \in P_{\underline{n}}$ with $m \in L_{\underline{n}}$ and n the unipotent part of p, we have

$$\lambda(p) = \lambda(m),$$

that is the unipotent part acts trivially. It is easy to check that this still gives group homomorphism $\lambda(p_1 p_2) = \lambda(p_1) \circ \lambda(p_2)$.

Then we define parabolically induced representation

$$\pi = \mathrm{Ind}_{P_{\underline{n}}}^G \lambda = \{f : G \to V | f(pg) = \Delta_{P_{\underline{n}}}(p)^{-1/2} \lambda(p) f(g),\ f \text{ smooth}\}.$$

Here we uses additional character $\Delta_{P_{\underline{n}}}(p)^{-1/2}$ for the following reason (see [2, 2.6]):

Lemma 5.1. *If λ is unitary, then so is $\pi = \mathrm{Ind}_{P_{\underline{n}}}^G \lambda$ with unitary structure given by*

$$< f_1, f_2 >_\pi = \int_{k \in K_p} < f_1(k), f_2(k) >_\lambda dk.$$

Remark 5.2. From the work of Bernstein–Zelevinsky, we know that

(1) π constructed from parabolic induction would be irreducible (automatically admissible) if $\lambda_i \not\simeq \lambda_j \otimes |\cdot|^{\pm 1}$. When it is not irreducible, one can also find and parameterize an irreducible subquotient representation of it. But we will not worry about this here.

(2) When $\mathbb{Q}_v = \mathbb{R}$, every representation can be constructed this way. But over $\mathbb{Q}_p$, there are representations which do not come from parabolic induction.

Definition 5.3. If there is no proper parabolic subgroup $P_{\underline{n}}$ and representation λ of $P_{\underline{n}}$ such that π is a subquotient of π_λ, then π is called a supercuspidal representation. This includes the case of character χ of GL_1.

With this definition, π is either a supercuspidal representation, or comes from parabolic induction.

(3) The order of λ_i doesn't matter. For example in case of being irreducible, we have

$$\mathrm{Ind}_P^G(\lambda_1 \otimes \lambda_2) \simeq \mathrm{Ind}_{P'}^G(\lambda_2 \otimes \lambda_1),$$

where P corresponds to $\mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}$, while P' corresponds to $\mathrm{GL}_{n_2} \times \mathrm{GL}_{n_1}$

Theorem 5.4. *There exists supercuspidal representation λ_i of GL_{n_i} such that π is a subquotient of parabolic induction $\mathrm{Ind}_{P_{\underline{n}}}^G(\lambda_1 \otimes \cdots \lambda_k)$. This includes the case $L_{\underline{n}} = GL_n$, where π is directly a supercuspidal representation.*

5.3. **Conductor of a principal series representation.** For simplicity, we consider the p-adic fields and the following

Definition 5.5. A parabolic induction $\mathrm{Ind}_P^G(\lambda)$ is called a principal series representation, if π is irreducible and $\lambda = \otimes\chi_1$ is 1-dimensional. In this case we also write $\pi = \boxplus\chi_i$.

Here we shall be interested in the conductor of a principal series representation.

Definition 5.6. The conductor $C(\pi)$ of a Good representation π of G is $p^{c(\pi)}$ where $c(\pi)$ is the smallest integer such that π admits a $K(p^{c(\pi)})$−invariant vector, where

$$K(p^{c(\pi)}) = \left\{ \begin{pmatrix} A & B \\ C & d \end{pmatrix} \mid A \text{ is rank } n-1 \text{ square matrix }, C \equiv 0 \mod p^{c(\pi)} \right\}.$$

The (up to scalar) $K(p^{c(\pi)})$−invariant vector is then called a newform.

Remark 5.7. Conductor also appears in the functional equation, but we shall skip here.

Lemma 5.8. *Let π be a principal series representation induced from characters χ_i, $i = 1, \cdots n$ over a p-adic field. Then $c(\pi) = \sum_i c(\chi_i)$ and $C(\pi) = p^{c(\pi)}$.*

We explain the result (rather than proving it) using local Langlands correspondence, which is more intuitive and avoids complicated computations, but its relation to the original definition is hidden.

The local Langlands correspondence predicts a corresponding of two types of representations

$$\{\text{Good representations of } \mathrm{GL}_n(\mathbb{F})\} \leftrightarrow \left\{ \begin{array}{c} n\text{−dimensional complex representations} \\ \text{of Weil–Deligne group } \mathrm{WD}_{\mathbb{F}} \end{array} \right\}.$$

with matching L−functions, gamma functions, and conductors. In the case $\pi = \boxplus\chi_i$, the corresponding Weil–Deligne representation is $LLC(\pi) = \oplus\chi_i'$ where each χ_i' is associated to χ_i via Class field theory (i.e., the local Langlands correspondence for GL_1). It is then nature to expect that

$$c(\pi) = \sum_i c(\chi_i') = \sum_i c(\chi_i).$$

An additional benefit for this perspective is that one can also understand the Rankin–Selberg product $\pi_1 \times \pi_2$. $\pi_1 \times \pi_2$ under (local) Langlands correspondence is related to $LLC(\pi_1) \otimes LLC(\pi_2)$. Further if $\pi_j = \boxplus \chi_{i,j}$, then

$$LLC(\pi_1 \times \pi_2) = LLC(\pi_1) \otimes LLC(\pi_2) = \oplus \chi'_{i,1}\chi'_{j,2}$$

and

$$C(\pi_1 \times \pi_2) = \prod_{i,j} C(\chi_{i,1}\chi_{j,2}).$$

Lemma 5.9. *Over a $p-$adic field, let π be a principal series representation induced from characters χ_i with $c(\chi_i) = 2k$ is an even integer, then there exists open compact subgroup*

$$J_\pi := \begin{pmatrix} O^\times & \mathfrak{p}^k & \cdots & \mathfrak{p}^k \\ \mathfrak{p}^k & O^\times & \cdots & \mathfrak{p}^k \\ \cdots & \cdots & \cdots & \cdots \\ \mathfrak{p}^k & \mathfrak{p}^k & \cdots & O^\times \end{pmatrix}$$

and character χ_π of J_π defined by

$$\chi_\pi((a_{ij})) = \prod_i \chi_i(a_{ii}),$$

such that there exists $v_\pi \in \pi$ with property that

$$\pi(j)v_\pi = \chi_\pi(j)v_\pi$$

for all $j \in J_\pi$.

Proof. We prove this result by explicitly constructing such an element in the parabolic induction model for π. We require $v_\pi \in \mathrm{Ind}_P^G \lambda$ to be supported on BJ_π, and on the support is defined by

$$v_\pi(bj) = \Delta_B^{-1/2}(b)\chi(b)\chi_\pi(j)v_\pi(1) \tag{5.1}$$

where $v(1)$ can be any element in $\mathbb{C}$ which is the space of representation for $\chi = \otimes\chi_i$. There are some ambiguity for the decomposition of matrix in BJ_π, that is if $x \in B \cap J_\pi$, there are two ways to write $v_\pi(x)$ and we need to check they are consistent:

$$v_\pi(x) = \Delta_B^{-1/2}(x)\chi(x)v_\pi(1) \qquad \text{or} \qquad \chi_\pi(x)v(1).$$

Indeed

$$B \cap J_\pi = \begin{pmatrix} O^\times & \mathfrak{p}^k & \cdots & \mathfrak{p}^k \\ 0 & O^\times & \cdots & \mathfrak{p}^k \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & O^\times \end{pmatrix}$$

so $\Delta_B^{-1/2}(x) = 1, \chi(x) = \chi_\pi(x) = \prod_i \chi_i(x_{ii})$.

One can then check that for v_π in (5.1), indeed $\pi(j)v_\pi = \chi_\pi(j)v_\pi$. □

Remark 5.10. The existence of v_π above is a more precise information than a newform. For example, the volume of J_π above is approximately

$$\frac{1}{p^{kn(n-1)}}.$$

On the other hand, $c(\pi) = 2kn$, and the volume of congruence subgroup $K_0(p^{c(\pi)})$ is

$$\frac{1}{p^{2kn(n-1)}}.$$

5.4. **Lie algebra description.** Consider for simplicity p large enough and $\pi = \boxplus \chi_i$ and $c(\chi_i) = 2k$. We know from GL_1 case that there exists $\alpha_i \in \mathcal{O}^\times$ such that

$$\chi_i(\exp(x)) = \psi(p^{-2k}\alpha_i x)$$

for all $x \in \mathfrak{p}$. Recall we also found $v_\pi \in \pi$ on which J_π acts by a character χ_π. Our goal is to describe this vector also in terms of Lie algebra.

Lemma 5.11. *For*

$$x \in \begin{pmatrix} \mathfrak{p} & \mathfrak{p}^k & \cdots & \mathfrak{p}^k \\ \mathfrak{p}^k & \mathfrak{p} & \cdots & \mathfrak{p}^k \\ \cdots & \cdots & \cdots & \cdots \\ \mathfrak{p}^k & \mathfrak{p}^k & \cdots & \mathfrak{p} \end{pmatrix}, \quad \alpha_\pi = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \alpha_n \end{pmatrix},$$

we have

$$\chi_\pi(\exp(x)) = \psi \circ Tr(p^{-2k}\alpha_\pi x).$$

Proof. We write $x = a + b$ for $a \in A$ diagonal, and $b \in p^k M_n(\mathcal{O})$ with $b_{ii} = 0$. Then

$$\begin{aligned} \exp(x) &= 1 + (a+b) + \frac{(a+b)^2}{2} + \cdots \\ &= \exp(a) + \sum_{i,j\geq 0} C_{i,j} a^i b a^j + O(p^{2k}) \end{aligned}$$

where all terms with at least two copies of b enter the $O(p^{2k})$ part. Note that $\sum_{i,j\geq 0} C_{i,j} a^i b a^j$ part still has vanishing diagonal entries. Error terms in $O(p^{2k})$ could have nonvanishing diagonal entries, but $c(\chi_i) = 2k$ so they doesn't matter. Then by definition of χ_π, we have

$$\chi_\pi(\exp(x)) = \chi_\pi(\exp(a)) = \prod \chi_i(\exp(a_{ii})) = \prod \psi(p^{-2k}\alpha_i a_{ii})$$

and on other hand

$$\psi \circ \mathrm{Tr}(p^{-2k}\alpha_\pi x) = \psi \circ \mathrm{Tr}(p^{-2k}\alpha_\pi a) = \prod \psi(p^{-2k}\alpha_i a_{ii}).$$

□

Corollary 5.12. *If χ_i are unitary characters with associated elements α_i as in Corollary 2.7 such that $\alpha_i \not\equiv \alpha_j$ when $i \neq j$, then the local matrix coefficient for v_π of above lemma satisfies the following property (after normalization):*

$$\Phi_{v_\pi}|_K(x) = \begin{cases} \chi_\pi(x), & \text{if } x \in J_\pi, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Recall that

$$\Phi_{v_\pi}(x) = <\pi(x)v_\pi, v_\pi> = \int_{k\in K} v_\pi(kx)\overline{v_\pi}(k)dk,$$

After normalizing $\Phi_{v_\pi}(1) = 1$, the values on J_π is clear.

We briefly explain why the pairing is vanishing when $x \in K - J_\pi$. Note that for it to be non-vanishing, it is necessary that for any $1 + y \in 1 + p^k M_n(\mathbb{Z}_p)$ we have

$$\begin{aligned} < \pi(1+y)\pi(x)v_\pi, \pi(1+y)v_\pi > &= \psi \circ \mathrm{Tr}(-p^{-2k}\alpha_\pi y) < \pi(x)\pi(1+x^{-1}yx)v_\pi, v_\pi > \\ &= \psi \circ \mathrm{Tr}(p^{-2k}\alpha_\pi[x^{-1}yx - y]) < \pi(x)v_\pi, v_\pi >, \\ &= \psi \circ \mathrm{Tr}(p^{-2k}[x\alpha_\pi x^{-1} - \alpha_\pi]y) < \pi(x)v_\pi, v_\pi >, \end{aligned}$$

which is equivalent to that $x\alpha_\pi x^{-1} - \alpha_\pi \equiv 0 \mod p^k$, which implies that x commutes with α_π mod p^k. As α_π is diagonal with $\alpha_i \not\equiv \alpha_j$, only such x must also be diagonal after modding p^k, which implies that $x \in J_\pi$.

□

Remark 5.13. Note that if J_π acts on v_π by χ_π, then for any $g \in G$, we have that $gJ_\pi g^{-1}$ acts on $\pi(g)v_\pi$ by character $\chi_\pi^g(gjg^{-1}) = \chi_\pi(j)$. We call all such $\pi(g)v_\pi$ localized vectors.

Using Lie algebra description, when α_π is associated χ_π, then $g\alpha_\pi g^{-1}$ is associated to χ_π^g, as

$$\chi_\pi^g(\exp(gxg^{-1})) = \chi_\pi(\exp(x)) = \psi \circ \mathrm{Tr}(p^{-2k}\alpha_\pi x) = \psi \circ \mathrm{Tr}(p^{-2k}g\alpha_\pi g^{-1}gxg^{-1}).$$

The orbit $\{g\alpha_\pi g^{-1}, g \in G\}$ is also called the coadjoint orbit associated to π. We shall take proper conjugation of α_π later on, though we will mainly use $g \in K_p$ so after conjugation,

$$gJ_\pi g^{-1} = gA(O^\times)g^{-1}(1 + p^k M_n(O)).$$

5.5. **A special supercuspidal representation case.** We shall not pursue the full story of supercuspidal representations here, only discuss a simpler family of them. Those interested can check out [4].

Given an element $\alpha \in M_n(\mathbb{F})$, suppose that its characteristic polynomial f_α is irreducible. Then we get a degree n field extension $\mathbb{L}$ over $\mathbb{F}$ generated by α together with embedding of $\mathbb{L}$ into $M_n(\mathbb{F})$:

$$\iota : \mathbb{L} \simeq \mathbb{F}[x]/(f_\alpha) \hookrightarrow M_n(\mathbb{F}).$$

Note that this embedding is consistent with traces, in the sense that

$$\mathrm{Tr}(\iota x) = \mathrm{Tr}_{\mathbb{L}/\mathbb{F}}(x)$$

for any $x \in \mathbb{L}$.

Example 5.14. Let $\alpha = \begin{pmatrix} 0 & 1 \\ D & 0 \end{pmatrix}$ with $D \in O^\times$ which is not a square. Then α generates a degree 2 inert field extension $\mathbb{L} = \mathbb{F}(\sqrt{D})$ over $\mathbb{F}$. Furthermore, α also generates the corresponding residue field of $\mathbb{L}$.

We further impose the following conditions for our special case:

Setting 1. (1) $\mathbb{L}$ is an inert/unramified field extension over $\mathbb{F}$ of degree n. So the residue field for $\mathbb{L}$ is also a degree n field extension over that of $\mathbb{F}$;
(2) $\alpha = p^{-2k}\alpha_\pi$ for some $\alpha_\pi \in O_{\mathbb{L}}^\times$, and we require that α_π generates the residue field of $\mathbb{L}$.

There exists a multiplicative character θ of $\mathbb{L}^\times$ such that

$$\theta(\exp(x)) = \psi \circ \mathrm{Tr}_{\mathbb{L}/\mathbb{F}}(p^{-2k}\alpha_\pi x) \tag{5.2}$$

As we can also change the embedding of $\mathbb{L}$ by a conjugation, we can, for example, fix the embedding so that α_π is in its rational form, that is, if $f_{\alpha_\pi} \in O[x]$ is the characteristic polynomial for α_π with $f_{\alpha_\pi} = x^n + a_1 x^{n-1} + \cdots + a_n$, then

$$\alpha_\pi \hookrightarrow \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{pmatrix}$$

Note that for this embedding, we have $\alpha_\pi \in K_p$ as $a_n \sim \det(\alpha_\pi) \in O^\times$. Thus α_π, and whole $\mathbb{L}^\times$ normalizes $1 + p^k M_n(O)$.

We define now a compact open subgroup

$$J_\pi = \mathbb{L}^\times(1 + p^k M_n(O)),$$

and a character of J_π

$$\chi_\pi(l\exp(x)) = \theta(l)\psi \circ \mathrm{Tr}(p^{-2k}\alpha_\pi x).$$

It is well defined due to (5.2). It is a character due to that $\mathbb{L}^\times$ normailzes $(1 + p^k M_n(O))$.

Lemma 5.15. *The representation*

$$\begin{aligned}\pi &= c - \mathrm{Ind}_{J_\pi}^G \chi_\pi \\ &= \{f : G \to \mathbb{C} | f(jg) = \chi_\pi(j)f(g), \forall j \in J_\pi, \text{ and } f \text{ is compactly supported mod center } \}\end{aligned}$$

is an irreducible (smooth) admissible supercuspidal representation. Its conductor is p^{2kn}.

When χ_π is unitary, π is also unitary with $G-$invariant pairing given by

$$< f_1, f_2 >= \int_{Z\backslash G} f_1(g)\overline{f_2}(g)dg.$$

Indeed the $G-$invariance comes from a change of variable, and it's absolutely convergent by that f_i have compact support mod center.

Corollary 5.16. *For π constructed above, there exists $v_\pi \in \pi$ such that J_π acts on v_π by χ_π. Its matrix coefficient satisfies*

$$\Phi_{v_\pi}(g) = \begin{cases} \chi_\pi(g), & \text{if } g \in J_\pi; \\ 0, & \text{otherise.} \end{cases}$$

Similar to the principal series representation case, when p is large enough, we have Lie algebra description of χ_π:

$$\chi_\pi(\exp(x)) = \psi \circ \mathrm{Tr}(p^{-2k}\alpha_\pi x)$$

for any $x \in \mathfrak{p}_{\mathbb{L}} + p^k M_n(O)$. We can also conjugate around the datum $(J_\pi, \chi_\pi, \alpha_\pi)$ as in principal series representation case.

6. Global tools

6.1. **Eisenstein series.** Our first goal is the spectral decomposition for automorphic forms, but for that purpose we need to introduce the Eisenstein series. They are also connected to parabolic inductions. For simplicity we consider $G = \mathrm{GL}_2$ case. Let $\chi_i, i = 1, 2$ be characters of $\mathbb{Q}^\times\backslash\mathbb{A}^\times$, $\chi_i = \prod \chi_{i,v}$. At each local place, we construct for a parameter $s \in \mathbb{C}$

$$\pi_{s,v} = \mathrm{Ind}_B^G(\chi_{1,v}|\cdot|_v^s, \chi_{2,v}|\cdot|_v^{-s})$$

we pick local component $\phi_{s,v} \in \pi_{s,v}$ so that $\phi_{v,s}|_{K_p}$ is independent of s and is constant 1 for a.a. p. Then we obtain an element $\phi_s = \prod \phi_{s,v} \in \pi_s = \prod{}' \pi_{s,v}$.

Note that ϕ_s is **not** an automorphic form, as it only has left $B(\mathbb{Q})-$invariance: indeed it is left $N(\mathbb{Q})-$invariant by parabolic induction, and left $A(\mathbb{Q})-$invariant by invariance property of χ_i.

To force $G(\mathbb{Q})-$invariance , we define first for $\mathrm{Re}(s)$ large enough

$$E(\phi, s)(g) = \sum_{\gamma \in B(\mathbb{Q}) \backslash G(\mathbb{Q})} \phi_s(\gamma g).$$

which is convergent and is now an automorphic form. One can prove that it has analytic continuation to $\mathbb{C}$ and satisfies functional equation. It could have pole at, for example, $s = 1$ with residue also an automorphic form.

In general for a bunch of cusp forms on Levi subgroup, one can form similar Eisenstein series. This is the direct generalization of classical Eisenstein series on upper half place.

6.2. **Spectral decomposition.** Again we assume for simplicity trivial central character.

Theorem 6.1. *For any $f \in L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$, we have*

$$f(g) = \sum_{\pi} \sum_{\varphi \in \mathcal{B}(\pi)} < f, \varphi > \varphi(g) + \cdots .$$

Here the sum in π is over global irreducible cuspidal (smooth) admissible automorphic representations, $\mathcal{B}(\pi)$ is an ONB for π.

$\cdots$ part represents sum over residue spectrum (being residue at poles of Eisenstein series) and integral over Eisenstein series. The cuspidal+residue parts are also called discrete spectrum and are analogue of Fourier expansion for $L^2([0,1])$, while the Eisenstein series part is called continuous spectrum and is analogue of Fourier expansion for $L^2(\mathbb{R})$. We shall not worry about the details here.

Remark 6.2. This result reduces arbitrary automorphic form to those from π which is factorisable. Implicit in this decomposition is a multiplicity one result, which means that each cuspidal irreducible π appears in $L^2[\mathrm{PG}]$ once.

6.3. **Period integrals.** (For more examples of period integrals on GL_n see [5, Lecture 5].) In our context, period integrals refer to integrals of automorphic forms over global domain, which can be further related to factorisable integrals and L-functions. To name a few: integral as in Tate's theis, Waldspurger's period integral, Rankin–Selberg integral, triple product integral, those of Gan–Gross–Prasad. Here we are concerned about Rankin–Selberg for $(G, H) = (\mathrm{GL}_{n+1}, \mathrm{GL}_n)$, or Gan–Gross–Prasad for $(G, H) = (U_{n+1}, U_n)$. In both cases, H can be embedded into G as $\begin{pmatrix} * & \\ & 1 \end{pmatrix}$. We use this embedding in the following.

Theorem 6.3. *Let $\varphi \in \pi$ be a cuspidal automorphic form for GL_{n+1} and $\phi \in \sigma$ be a cusp form for GL_n. Then*

$$I := \int_{h \in [GL_n]} \varphi(h)\phi(h) \| \det(h) \|^{s-1/2} dh \sim L(\pi \times \sigma, s) \prod_{v \in S} I_v$$

where $\| \cdot \| = \prod | \cdot |_v$ is the adelic absolute value, S is a finite set of places including ∞ and every places where π_v or σ_v are not unramified (i.e., not principal series representation induced from *unramified characters). $\sim$ means equality up to some not important factors. I_v is the local integral given by*

$$I_v \sim \int_{N_n(\mathbb{Q}_v)\backslash GL_n(\mathbb{Q}_v)} W_{\varphi,v}(h)W^-_{\phi,v}(h)|\det(h)|_v^{s-1/2}dh.$$

Here W^- denote the Whittaker function associated to $\psi^-(x) = \psi(-x)$.

Partial proof. Recall Lemma 4.20 for Whittaker function and Whittaker expansion in general.

Start with Whittaker expansion for φ and $\mathrm{Re}(s)$ large enough, we get

$$\begin{aligned} I &= \int_{h\in[\mathrm{GL}_n]} \sum_{N_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{Q})} W_\varphi(\gamma h)\phi(h)\|\det(h)\|^{s-1/2}dh \\ &= \int_{h\in N_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{A})} W_\varphi(h)\phi(h)\|\det(h)\|^{s-1/2}dh \\ &= \int_{h\in N_n(\mathbb{A})\backslash \mathrm{GL}_n(\mathbb{A})} \int_{u\in[N_n]} W_\varphi(uh)\phi(uh)\|\det(uh)\|^{s-1/2}dudh \\ &= \int_{h\in N_n(\mathbb{A})\backslash \mathrm{GL}_n(\mathbb{A})} W_\varphi(h) \int_{u\in[N_n]} \phi(uh)\psi_0(u)du\|\det(h)\|^{s-1/2}dh \\ &= \int_{h\in N_n(\mathbb{A})\backslash \mathrm{GL}_n(\mathbb{A})} W_\varphi(h)W^-_\phi(h)\|\det(h)\|^{s-1/2}dh \end{aligned}$$

Here we have used that ψ_0 of N_{n+1} and ψ_0 of N_n coincide upon restriction. The last line now factorizes into a product of local integrals like I_v. One can then compute explicitly that I_v gives the local L-function $L_v(\pi\times\sigma, s)$, which we shall skip here.

With analytic continuation and taking $s = 1/2$, we get

$$\int \varphi(h)\phi(h)dh \sim L(\pi\times\sigma, 1/2)\prod_{v\in S} I_v$$

where

$$I_v \sim \int_{N_n(\mathbb{Q}_v)\backslash \mathrm{GL}_n(\mathbb{Q}_v)} W_{\varphi,v}(h)W^-_{\phi,v}(h)dh.$$

□

We also record here the Gan–Gross–Prasad case. (See the arXiv paper by Plessis–Chaudouard–Zydor)

Theorem 6.4. *Let $\varphi \in \pi$ be an automorphic form on $[U_{n+1}]$ and $\phi \in \sigma$ on $[U_n]$. Then*

$$|\int_{[H]} \varphi(h)\phi(h)dh|^2 \sim \mathcal{L}(\pi,\sigma)\prod_{v\in S}\mathcal{I}_v$$

where

$$\mathcal{I}_v = \int_{h\in H(\mathbb{Q}_v)} \Phi_{\varphi,v}(h)\Phi_{\phi,v}(h)dh.$$

Here one should roughly think of $\mathcal{L}(\pi, \sigma)$ as $|L(\pi \times \sigma, 1/2)|^2$, and $\mathcal{I}_v$ roughly as $|I_v|^2$. Then this is an exact analogue of $\mathrm{GL}_{n+1} \times \mathrm{GL}_n$ case.

7. RELATIVE TRACE FORMULA

Here we give an introduction to trace formulae, without full justifications

7.1. **pretrace formula.** Given $f \in S(\mathrm{PG}(\mathbb{A}))$ Schwartz function such that $f = \otimes f_v$ with $f_p = \mathrm{char}(G(\mathbb{Z}_p))$ for a.a. p, we define an integral operator on $L^2([\mathrm{PG}])$

$$\rho(f) : \varphi \to \int\limits_{\mathrm{PG}(\mathbb{A})} \varphi(xg) f(g) dg$$

This operator can be represented by a so-called integral kernel $K_f(x, y)$, in the sense that

$$\rho(f)\varphi(x) = \int\limits_{[\mathrm{PG}]} K_f(x, y)\varphi(y)dy$$

We now use two ways to express $K_f(x, y)$, which will give an equality.

7.1.1. *Via integral manipulations.* From definition and change of variable we have

$$\begin{aligned}
\rho(f)\varphi(x) &= \int\limits_{\mathrm{PG}(\mathbb{A})} \varphi(g) f(x^{-1}g) dg \\
&= \int\limits_{y \in [\mathrm{PG}]} \sum_{\gamma \in \mathrm{PG}(\mathbb{Q})} \varphi(\gamma y) f(x^{-1}\gamma y) dy \\
&= \int\limits_{y \in [\mathrm{PG}]} \left(\sum_{\gamma \in \mathrm{PG}(\mathbb{Q})} f(x^{-1}\gamma y) \right) \varphi(y) dy
\end{aligned}$$

Thus we obtain first the 'geometric' expression of $K_f(x, y)$:

$$K_f(x, y) = \sum_{\gamma \in \mathrm{PG}(\mathbb{Q})} f(x^{-1}\gamma y).$$

7.1.2. *Via spectral decomposition.* On the other hand, we can view $\overline{K_f(x, y)}$ as an automorphic form in variable y. Indeed it is left $\mathrm{PG}(\mathbb{Q})-$invariant from the above geometric expression. Applying spectral decomposition for this form in x, we get

$$\overline{K_f(x, y)} = \sum_{\pi} \sum_{\varphi \in \mathcal{B}(\pi)} < \overline{K_f(x, y)}, \varphi > \varphi(y) + \cdots$$

where

$$< \overline{K_f(x, y)}, \varphi >= \int\limits_{[\mathrm{PG}]} \overline{K_f(x, y)\varphi(y)} dy = \overline{\rho(f)\varphi(x)}.$$

Taking complex conjugate of above spectral decomposition, we get

$$K_f(x, y) = \sum_{\pi} \sum_{\varphi \in \mathcal{B}(\pi)} \rho(f)\varphi(x)\overline{\varphi(y)} + \cdots .$$

Combining the two ways to representation $K_f(x, y)$,we get

Theorem 7.1 (Pretrace formula)**.**

$$\sum_{\pi}\sum_{\varphi\in\mathcal{B}(\pi)}\rho(f)\varphi(x)\overline{\varphi(y)}+\cdots=\sum_{\gamma\in PG(\mathbb{Q})}f(x^{-1}\gamma y)$$

Remark 7.2. We will be mostly interested in the case $\rho(f)$ is a projection operator, that is when

$$\rho(f)=\rho(f)\circ\rho(f)=\rho(f*f).$$

In that case we can rewrite spectral side of pretrace formula as

$$\sum_{\pi}\sum_{\varphi\in\mathcal{B}(\mathrm{Im}(\rho(f)))}\varphi(x)\overline{\varphi(y)}+\cdots$$

7.2. **relative trace formula.** From pretrace formula, one can integrate each variable of the kernel function $K_f(x,y)$ against other functions on possibly different domains.

Example 7.3. For $G=\mathrm{GL}_2$, we can consider

$$\iint_{x,y\in[N]}K_f(x,y)\psi(-x)\psi(y)dxdy.$$

Then on the spectral side, we will get Whittaker function/Fourier coefficients. The geometric on the other side can be related to Kloosterman sum. This relative trace formula can recover the classical Peterrson/Kuznetsov trace formula.

Now we take $(G,H)=(\mathrm{GL}_{n+1},\mathrm{GL}_n)$ or (U_{n+1},U_n). Let $\rho(f)$ be a projection operator for automorphic forms on G. Let σ be a cuspidal automorphic representation of H. Choosing proper $\phi\in\sigma$, we consider the following relative trace formula

Theorem 7.4 (Relative trace formula for pair (G,H))**.**

$$\begin{aligned}\int_{x,y\in[H]}K_f(x,y)\phi(x)\overline{\phi(y)}dxdy&=\sum_{\pi}\sum_{\varphi\in\mathcal{B}(\mathrm{Im}(\rho(f)))}\int_{[H]}\varphi(x)\phi(x)dx\overline{\int_{[H]}\varphi(y)\phi(y)dy}+\cdots\\&=\iint_{x,y\in[H]}\sum_{\gamma\in PG(\mathbb{Q})}f(x^{-1}\gamma y)\phi(x)\overline{\phi(y)}dxdy\end{aligned}$$

The spectral side in the first line can further be related to $L(\pi\times\sigma,1/2)$ *or* $\mathcal{L}(\pi,\sigma)$ *and local integrals by period integral formula.*

8. Application to subconvexity bound

We can now give an outline to obtain subconvexity bound for $L(\pi\times\sigma,1/2)$ or $\mathcal{L}(\pi,\sigma)$ via applying relative trace formula and choosing proper localized vectors as in Remark 5.13.

We shall be mainly focus on the case $(G,H)=(U_{n+1},U_n)$ with trivial central characters for π and σ, which allows the following simplifications compared with GL case.

(1) For the local period integral on the spectral side of relative trace formula, we consider the matrix coefficient version

$$\mathcal{I}_v=\int_{H(\mathbb{Q}_v)}\Phi_{\varphi,v}(h)\Phi_{\phi,v}(h)dh,$$

so we can apply known description of matrix coefficient for localized vectors. It is also possible to describe their Whittaker functions, but that requires additional efforts.

(2) For the geometric side the x, y variables belong to a compact domain of $[H]$, which is true for U_n case. This greatly simplifies the analysis for geometric side. Landing inside a non-compact domain is actually a big challenge for the GL_n case.

We restrict to depth aspect and specify the details now:

Setting 2. (1) We assume $\pi_\infty, \sigma_\infty$ are fixed or bounded. π_l, σ_l are unramified at all finite places except at a place p, which is large enough (compared with n) but fixed.

(2) At p we assume for simplicity that σ_p is a principal series representation induced from χ_i with $c(\chi_i) = 2k$ and $C(\sigma_p) = p^{2kn}$. Assume that π_p is a special supercuspidal representation as in Section 5.5 with conductor $C(\pi_p) = p^{2k(n+1)}$.

Our goal is to partially explain why the following subconvexity result is plausible:

Theorem 8.1. *For (G, H) and (π, σ) as above, we have*

$$\mathcal{L}(\pi, \sigma) \ll C(\pi, \sigma)^{1/4-\delta}$$

for some $\delta > 0$.

8.1. **Conductor and goal of subconvexity bound.**

Lemma 8.2. *As in Setting 2 we have $C_p(\pi \times \sigma) = p^{2kn(n+1)}$. This means that our goal for subconvexity is*

$$\mathcal{L}(\pi, \sigma) \ll p^{(n(n+1)-\delta)k}$$

for some $\delta > 0$ for the depth aspect when we let p be fixed and $k \to \infty$.

Proof. Similar to principal series representation case, we have

$$C_p(\pi \times \sigma) = \prod C_p(\sigma \times \chi_i) = \prod p^{2k(n+1)} = p^{2kn(n+1)}.$$

$C(\pi, \sigma)$ is asymptotically $C_p(\pi \times \sigma)^2$. Hence the second part of lemma. □

Remark 8.3. For Setting 2(2), we automatically have that π, σ is not in conductor dropping range, that is $C(\pi \times \sigma)$ is as large as expected. If we assume π_p is also a principal series representation induced from characters η_j with $c(\eta_j) = 2k$, then

$$C(\pi \times \sigma) = p^{\sum\limits_{i,j} c(\chi_i \eta_j)}.$$

Then we need all $c(\chi_i \eta_j) = 2k$ as well to not be in the conductor dropping range. Otherwise $C(\pi \times \sigma)$ becomes smaller than expected, meaning that the required subconvexity bound is more difficult and not available for our current method.

Being able to deal with conductor dropping range is actually very challenging and is an open problem.

8.2. **Choice of test vectors.** We specify now the local test vectors $\varphi_p \in \pi_p$, $\phi_p \in \sigma_p$, so that we get the proper size for the local period integral on the spectral side.

Recall that in $\sigma = \mathrm{Ind}_B^H(\chi_1, \cdots, \chi_n)$ with $c(\chi_i) = 2k$, we can find localized test vector on which J_σ acts by a character χ_σ, which can be represented by an element α_σ in proper neighborhood, and we can conjugate all data. We pick ϕ_p to be a localized vector associated to $g\alpha_\pi g^{-1}$ is in the following form

$$\alpha'_\sigma = \begin{pmatrix} -a_1 & 1 & \cdots & 0 & 0 \\ -a_2 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_{n-1} & 0 & \cdots & 0 & 1 \\ -a_n & 0 & \cdots & 0 & 0 \end{pmatrix}$$

which is just the rational form with order of basis reversed. One can also choose g that realizes the conjugation to be in $\mathrm{GL}_n(\mathbb{Z}_p)$, so that the resulting compact open subgroup has the shape

$$J'_\sigma = gAg^{-1}(1 + p^k M_n(O)).$$

Now for the supercuspidal representation π of GL_{n+1}, we use a localized vector φ_p associated to the element

$$\alpha'_\pi = \begin{pmatrix} a_1 & -1 & \cdots & 0 & 0 & 0 \\ a_2 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & 0 & \cdots & 0 & -1 & 0 \\ a_n & 0 & \cdots & 0 & 0 & -1 \\ -c_{n+1} & -c_n & \cdots & -c_3 & -c_2 & -c_1 \end{pmatrix}.$$

Here c_i are parameters to be determined so that the characteristic polynomial of α'_π is exactly that of α_π. This is a purely linear algebra problem, and is possible as there are $n+1$ parameters c_i to match $n+1$ coefficients of f_{α_π}.

Example 8.4. Consider the case $\mathrm{GL}_3 \times \mathrm{GL}_2$. Then $f_{\alpha_\sigma} = x^2 + a_1 x + a_2$.

$$f_{\alpha'_\pi} = \left| \begin{pmatrix} x - a_1 & 1 & 0 \\ -a_2 & x & 1 \\ c_3 & c_2 & x + c_1 \end{pmatrix} \right| = x^3 + (-a_1 + c_1)x^2 + (-c_2 - a_1 c_1 + a_2)x + a_1 c_2 + a_2 c_1 + c_3.$$

From this one can see that for any target coefficients of f_{α_π} one can solve for c_1, c_2, c_3 inductively.

Similar to J'_σ, one can make the conjugation so that

$$J'_\pi = \mathbb{L}^{\times\prime}(1 + p^k M_{n+1}(O)).$$

Lemma 8.5. *For localized test vectors φ_p, ϕ_p specified as above, we have*

$$\mathcal{I}_p = \int_{H(\mathbb{Q}_p)} \Phi_{\varphi,p}(h)\Phi_{\phi,p}(h)dh \asymp \frac{1}{p^{kn^2}}.$$

sketch of proof. Recall from Corollary 5.16 that the matrix coefficient of $\Phi_{\varphi,p}$ for localized vector in supercuspidal representation is supported only on J'_π. We claim that

$$J'_\pi \cap H = 1 + p^k M_n(O).$$

Here we give partial reason. Indeed as noted above $J'_\pi = \mathbb{L}^{\times\prime}(1 + p^k M_{n+1}(O))$, and $\mathbb{L}^{\times\prime}$ should intersect with H trivially. This is because the characteristic polynomial of elements in $H \to G$ is of form $f(x) = (x-1)f_H(x)$ for f_H degree n. The characteristic polynomial for $l \in \mathbb{L}^{\times\prime}$ should be a power of an irreducible polynomial corresponding to an intermediate field of $\mathbb{L}$. From this we then expect

$$J'_\pi \cap H = (1 + p^k M_{n+1}(O)) \cap H = 1 + p^k M_n(O).$$

Now this intersection is in a small enough so we can apply Lie algebra description for both matrix coefficients:

$$
\begin{aligned}
\mathcal{I}_p &= \int\limits_{H(\mathbb{Q}_p)} \Phi_{\varphi,p}(h)\Phi_{\phi,p}(h)dh \\
&= \int\limits_{x\in p^k M_n(O)} \psi\circ \mathrm{Tr}(p^{-2k}\alpha'_\pi x)\psi\circ \mathrm{Tr}(p^{-2k}\alpha'_\sigma x)dx \\
&= \int\limits_{x\in p^k M_n(O)} \psi\circ \mathrm{Tr}(p^{-2k}(\alpha'_\pi + \alpha'_\sigma)x)dx.
\end{aligned} \tag{8.1}
$$

Now we see the reason we pick particular α'_σ and α'_π: we have

$$
\alpha'_\pi + \alpha'_\sigma = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -1 \\ -c_{n+1} & -c_n & \cdots & -c_3 & -c_2 & -c_1 \end{pmatrix}
$$

whose trace pairing with $x \in M_n(\mathbb{F}) \to M_{n+1}(\mathbb{F})$ is always 0. Thus the integrand in last line of (8.1) is constant 1 and

$$
\mathcal{I}_p \asymp \mathrm{Vol}(p^k M_n(O)) \asymp \frac{1}{p^{kn^2}}.
$$

□

8.3. **Choice of integral operator.** We can now define the function $f \in S(\mathrm{PG}(\mathbb{A}))$ for the integral operator in relative trace formula. We fix the choice of f_∞ which is compactly supported in $\mathrm{PG}(\mathbb{R})$. We choose $f_l = \mathrm{charPG}(\mathbb{Z}_l)$ for all $l \neq p$. The only changing thing is f_p.

Recall that we need $\rho(f)$ to be a projection operator, and we want φ_p specified in Section 8.2 to be in the image of this projection.

Definition 8.6. Let

$$
f_p(g) = \frac{1}{\mathrm{Vol}(J_\pi)} \begin{cases} \overline{\chi'_\pi}, & \text{if } g \in J'_\pi \\ 0, & \text{otherwise} \end{cases}.
$$

Note that

$$
\mathrm{Vol}(J_\pi) \asymp \frac{1}{p^{kn(n+1)}}.
$$

Lemma 8.7. *$\rho(f_p)$ is a projection operator with $\rho(f_p)\varphi_p = \varphi_p$.*

Proof. We check directly that

$$
f_p * f_p(x) = \frac{1}{\mathrm{Vol}(J_\pi)^2} \int\limits_{y\in J'_\pi} f_p(y) f_p(xy^{-1})dy
$$

For xy^{-1} to be in support, it is necessary that $x \in J'_\pi$, in which case we have

$$f_p * f_p(x) = \frac{1}{\mathrm{Vol}(J_\pi)^2} \int_{y \in J'_\pi} \overline{\chi'_\pi(y)}\,\overline{\chi'_\pi(xy^{-1})} dy$$
$$= \frac{1}{\mathrm{Vol}(J_\pi)} \overline{\chi'_\pi(x)} = f_p.$$

One can also easily compute that

$$\rho(f_p)\varphi_p = \frac{1}{\mathrm{Vol}(J_\pi)} \int_{y \in J'_\pi} \overline{\chi'_\pi(y)} \pi_p(y)\varphi_p dy$$
$$= \frac{1}{\mathrm{Vol}(J_\pi)} \varphi_p \int_{y \in J'_\pi} \overline{\chi'_\pi(y)} \chi'_\pi(y) dy = \varphi_p.$$

□

8.4. **Geometric side: main term and progress so far.** Recall that the geometric side of relative trace formula consists of

$$\iint_{x,y \in [H]} \sum_{\gamma \in \mathrm{PG}(\mathbb{Q})} f(x^{-1}\gamma y)\phi(x)\overline{\phi(y)} dx dy.$$

Among all $\mathrm{PG}(\mathbb{Q})$ are the embedded image of $H(\mathbb{Q})$, which we expect to be the main term of the geometric side. One can unfold the integral for main term as follows:

$$\mathrm{MT} := \iint_{x,y \in [H]} \sum_{\gamma \in H(\mathbb{Q})} f(x^{-1}\gamma y)\phi(x)\overline{\phi(y)} dx dy = \int_{x \in [H]} \int_{y \in H(\mathbb{A})} f(x^{-1}y)\phi(x)\overline{\phi(y)} dx dy$$
$$= \int_{x \in [H]} \int_{z \in H(\mathbb{A})} f(z^{-1})\phi(x)\overline{\phi(xz^{-1})} dx dz$$
$$= \int_{x \in [H]} \int_{z \in H(\mathbb{A})} f(z^{-1})\phi(xz)\overline{\phi(x)} dx dz = \int_{z \in H(\mathbb{A})} f(z^{-1})\Phi_\phi(z) dz.$$

Here in the second line we made a change of variable $y = xz^{-1}$. The last integral is factorisable, with components at ∞ and $l \neq p$ essentially fixed. So the estimation of the main term reduces to

$$\mathrm{MT}_p = \int_{z \in H(\mathbb{Q}_p)} f_p(z^{-1})\Phi_{\phi,p}(z) dz.$$

Substitute in the choice of local test function f_p in Definition 8.6, we get that

$$\mathrm{MT}_p = \frac{1}{\mathrm{Vol}(J_\pi)} \int_{H(\mathbb{Q}_p) \cap J'_\pi} \chi'_\pi(z)\Phi_{\phi,p}(z) dz$$

The constant $\mathrm{Vol}(J_\pi) \asymp \frac{1}{p^{kn(n+1)}}$, the integral itself is directly related to the integral of local matrix coefficient evaluated in Lemma 8.5. Thus

$$\mathrm{MT}_p \asymp p^{kn(n+1)} \frac{1}{p^{kn^2}} = p^{kn}.$$

We explain now that the main term alone gives exactly the convexity bound for $L-$function. For the spectral side we substitute in the size of local period integral, and drop all except one term (as we expect all terms are positive), we get

$$|L(\pi\times\sigma,1/2)|^2\frac{1}{p^{kn^2}}\leq \mathrm{MT}+\mathrm{ET}.$$

Ignoring the error terms for a moment, and put in the estimation of main term, we get

$$|L(\pi\times\sigma,1/2)|^2\leq p^{kn(n+1)}$$

which exactly match the convexity bound discussed in Lemma 8.2.

Our last step is to show that the remaining contribution from $\mathrm{PG}(\mathbb{Q})-H(\mathbb{Q})$ indeed gives error term, smaller than the main term by a power of p^k.

Remark 8.8. Of course this is not enough to get subconvexity. But it is a common situation that when the main term gives exactly the convexity bound, and error term is smaller, one can use the so-called amplification method to get a subconvexity bound. Roughly speaking, this method can balance the sizes of main term and error term, making main term smaller while error term larger.

8.5. **Geometric side: error term.** Here we make critical use of simplification that we consider only a compact domain of $[H]$. Our strategy to control the error terms is straightforward: count the number of γ and control individual terms in

$$\iint_{x,y\in[H]}\sum_{\gamma\in\mathrm{PG}(\mathbb{Q})-H(\mathbb{Q})}f(x^{-1}\gamma y)\phi(x)\overline{\phi(y)}dxdy.$$

8.5.1. *Counting* $\sharp\gamma$.

Lemma 8.9. *Let x,y be in a fixed compact region of* $[H]$. *then*

$$\sharp\{\gamma|f(x^{-1}\gamma y)\neq 0\}$$

is absolutely bounded

Proof. Recall that f_∞ is supported on some fixed bounded set of $\mathrm{PG}(\mathbb{R})$, $f_l=\mathrm{charPG}(\mathbb{Z}_l)$ for all $l\neq p$, f_p is more complicated, but one can also cover the support by $\mathrm{charPG}(\mathbb{Z}_p)$. x,y are in a compact region. Using approximation result, we can assume that $x_l,y_l\in\mathrm{PG}(\mathbb{Z}_l)$, and x_∞,y_∞ belong to a compact region of $\mathrm{PG}(\mathbb{R})$.

Let $\gamma\in\mathrm{PG}(\mathbb{Q})$ be such that $f(x^{-1}\gamma y)\neq 0$ for some x,y in the compact region. We have to be a little careful about the central direction.

The requirement at l implies that for any p-adic place l there exists a local center element $z_l\in\mathbb{Q}_l^\times$ such that $z_l\gamma\in\mathbb{G}(\mathbb{Z}_p)$. Using approximation for $\mathbb{A}^\times$, there exists $z\in\mathbb{Q}^\times$ with $v_l(z)=v_l(z_l)$. Then we also have $z\gamma\in\mathbb{G}(\mathbb{Z}_p)$. This implies that $\gamma_0:=z\gamma$ (note that γ_0 is identified with γ in $\mathrm{PG}(\mathbb{Q})$) is actually an integer matrix as all its entries have to be $l-$adically integral for all l. Furthermore it is also necessary that $\det(\gamma_0)\in\mathbb{Z}_l^\times$ for all l, implying that $\det(\gamma_0)=\pm1$.

On the other hand at ∞, the conditions on f_∞ and x_∞,y_∞ implies that there exists z_∞ such that $z_\infty\gamma_0$ lands in a fixed compact region Ω of $G(\mathbb{R})$, and so does $\det(z_\infty\gamma_0)=z_\infty^{n+1}\det(\gamma_0)$. From $\det(\gamma_0)=\pm1$, we get that z_∞ itself must be bounded, thus all entries of γ_0 are in some bounded region of $\mathbb{R}$. There are only finitely many such integer matrices γ_0.

□

Remark 8.10. This counting result becomes much more complicated when amplification method and non-compact region are involved.

8.5.2. *Control integral for individual* γ. For each individual γ such that $f(x^{-1}\gamma y) \neq 0$ for some x, y in the compact region of $[H]$, we have

$$|\iint_{x,y\in[H]} f(x^{-1}\gamma y)\phi(x)\overline{\phi(y)}dxdy|^2 \leq \iint |\phi|^2(x)|f|(x^{-1}\gamma y)dxdy \iint |\phi|^2(y)|f|(x^{-1}\gamma y)dxdy$$

By symmetry we look only at

$$I := \iint |\phi|^2(x)|f|(x^{-1}\gamma y)dxdy.$$

Recall that f_∞ f_l are fixed, and

$$|f_p| = \frac{1}{\mathrm{Vol}(J_\pi)}\mathrm{char}(\mathbb{L}^{\times\prime}(1 + p^k M_{n+1}(\mathbb{Z}_p))).$$

By choosing f carefully, we may assume without loss of generality, that supp$f$ is closed under taking inverse matrix.

We integrate in y first. For fixed $x \in [H]$, we first need $\gamma^{-1}x \in H\mathrm{supp}f$ to be able to find y such that $x^{-1}\gamma y \in \mathrm{supp}f$. Then for any two such y_i, we need $y_1 y_2^{-1} \in \mathrm{supp}f$. This implies that

$$\mathrm{Vol}(\{y \in [H], x^{-1}\gamma y \in \mathrm{supp}f\}) \begin{cases} \asymp \mathrm{Vol}(\mathrm{supp}f \cap [H]), & \text{if } \gamma^{-1}x \in H\mathrm{supp}f \\ = 0, & \text{otherwise.} \end{cases}$$

From the local computation of $\mathrm{Vol}(J'_\pi \cap H(\mathbb{Q}_p))$ in the proof of Lemma 8.5 (by picking a fundamental domain) and Lemma 8.9,

$$\mathrm{Vol}(\mathrm{supp}f \cap [H]) \asymp \frac{1}{p^{kn^2}}.$$

Then we have

$$I = \frac{p^{kn(n+1)}}{p^{kn^2}} \int_x |\phi|^2(x)\mathrm{char}(\gamma H\mathrm{supp}f)(x)dx \tag{8.2}$$

Here the constant multiple comes from size of f and volume estimates for supp$f \cap [H]$. At this point if we just bound the integral in (8.2) by the L^2-norm of ϕ, we get exactly the same size as the main term. So we need a nontrivial saving for

$$I' := \int_x |\phi|^2(x)\mathrm{char}(\gamma H\mathrm{supp}f)(x)dx$$

To achieve that we make use of the symmetry that at p J'_σ acts on ϕ_p by χ'_σ. Thus

$$\begin{aligned} I' &= \frac{1}{\mathrm{Vol}(J'_\sigma)} \int_{h\in J'_\sigma} \int_{x\in[H]} |\phi|^2(xh^{-1})\mathrm{char}(\gamma H\mathrm{supp}f)(x)dxdh \\ &= \int_{x\in[H]} |\phi|^2(x)\frac{1}{\mathrm{Vol}(J'_\sigma)} \int_{h\in J'_\sigma} \mathrm{char}(\gamma H\mathrm{supp}f)(xh)dhdx \end{aligned}$$

The problem can be proved using the following local result

Lemma 8.11. *Let $n \geq 2$ and $p > 3$. For any fixed x, γ,*

$$\{h \in J'_\sigma | gh \in H(\mathbb{Q}_p)J'_\pi\} \neq J'_\sigma$$

unless $g \in H(\mathbb{Q}_p)Z(\mathbb{Q}_p)$.

The proof of this result is not simple, but it is purely about local matrix algebra. We skip the proof here.

Remark 8.12. When they two sets are not equal, we get further a power saving as left hand side is actually an algebraic subvariety. This is roughly equivalent to that when we have a nontrivial polynomial equation, the number of solution is absolutely bounded in terms of degree of polynomial.

Course information

Title: Subconvexity problem on higher rank groups

Abstract: In this mini-course our goal is to explain the recent approach by Paul Nelson to get subconvexity bound for L-function on higher rank groups like U_n or GL_n. We will give some numerical evidence why this approach can give nontrivial bound. For this goal we shall also review some notions and tools in the theory of automorphic forms. The main topics include

(1) Local fields, adele, idele
(2) Basics about groups
(3) From modular forms to automorphic forms and representations
(4) Classification and examples of local representations
(5) Spectral decomposition and period integrals
(6) Relative trace formula
(7) Application to subconvexity bound

Prerequisites: Being familiar with algebraic number theory, modular forms, basics of representation theory. I will try to review most of necessary definitions and results in the course.

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