Homework 1

(due Wednesday, March 17, 2021)

1. Let $k \in \mathbb{N}$. Show that $(n-1)^2 \mid (n^k - 1)$ if and only if $(n-1) \mid k$.
2. Show that if $n > 1$, then $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is not an integer.
3. Let $a, m, n \in \mathbb{N}$. Show that if $(m, n) = 1$, then
$$(a^{m-1} + a^{m-2} + \cdots + a + 1, a^{n-1} + a^{n-2} + \cdots + a + 1) = 1.$$
4. Show that $(a, b, c)[a, b, c] = |abc|$ if and only if $(a, b) = (b, c) = (c, a) = 1$.
5. Show that there are infinitely many prime numbers of the form $6k + 5$.
6. Let $n \in \mathbb{N}$ and $a \geq 0$, then
   (i) $\left[\frac{[na]}{n}\right] = [a]$;
   (ii) $\sum_{k=0}^{n-1} [a + \frac{k}{n}] = [na]$.
7. Let $m, n \in \mathbb{N}$, show that $\frac{(2m)!(2n)!}{m!n!(m+n)!}$ and $\frac{(mn)!}{(n)!(m!)^n}$ are integers.
8. Show that
$$\sum_{d|n} \mu(d)\sigma(d) = (-1)^{\omega(n)} \prod_{p|n} p$$
and
$$\sum_{d|n} \mu(d)\varphi(d) = (-1)^{\omega(n)} \prod_{p|n} (p-2).$$
9. Show that if $n > 1$, then
$$\sum_{d|n} |\mu(d)| = 2^{\omega(n)}$$
and
$$\sum_{\ell|n} \mu(\ell) d(\ell) = (-1)^{\omega(n)}.$$
10. Show that
    (i) $\varphi(n) > \frac{1}{2}\sqrt{n}$;
    (ii) If n is a composite integer, we have $\varphi(n) \leq n - \sqrt{n}$.
11. Give all solutions of $\varphi(n) = 24$.
12. Give all $n \in \mathbb{N}$ such that $4 \nmid \varphi(n)$.