Spectral Theory for GL(2)

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§2.1 Adelization of Modular Forms

* Holomorphic M.F.s

Def 1. $J: GL(2,\mathbb{R})^+ \times \mathbb{H} \to \mathbb{C}$ is a factor of automorphy if

$$J(g_1g_2, z) = J(g_1, g_2.z)\, J(g_2, z) \quad \& \quad J(1, z) = 1$$

Classical factor of automorphy: $j(g,z) = (cz+d)\cdot|\det g|^{-\frac{1}{2}}$ for $g=\begin{pmatrix}a&b\\c&d\end{pmatrix}$

Def 2. For any $k \in \mathbb{Z}_{>0}$, the slash action of $GL_2(\mathbb{R})^+$ on $f: \mathbb{H} \to \mathbb{C}$

$$f|_{[g]_k}(z) := f(g.z)\, j(g,z)^{-k}$$

Def 3. For lattice $\Gamma < SL_2(\mathbb{R})$, $f \in M_k(\Gamma)$ for $f: \mathbb{H} \to \mathbb{C}$ iff

(1) $f|_{[\gamma]_k} = f$ for any $\gamma \in \Gamma$

(2) f is holomorphic on $\mathbb{H}$

(3) For any cusp s and some (all) $\alpha \in SL_2(\mathbb{R})$ s.t. $\alpha.\infty = s$, the function $f|_{[\alpha]_k}$ is bounded in a neighborhood of ∞

Moreover, $f \in S_k(\Gamma)$ if "bounded ... ∞" in (3) is replaced by "vanishes at ∞".

Lemma 1. Let $f \in M_k(\Gamma)$ and $\alpha \in SL_2(\mathbb{R})$ as in <u>Def 3 (3)</u>. Then $\exists h \in \mathbb{R}_{>0}$ s.t.

$$f|_{[\alpha]_k}(z) = \sum_{n=0}^{\infty} a_n e^{\frac{2\pi i n z}{h}}$$

And $f \in S_k(\Gamma) \iff a_0 = 0$ for all cusps.

Rmk: For $f_1, f_2 \in S_k(\Gamma)$, $g(z) := f_1(z)\overline{f_2(z)}(\mathrm{Im}\, z)^k$ satisfies $g(\gamma.z) = g(z)$, $\forall \gamma \in \Gamma$

Def 4. The hermitian form $\langle f_1, f_2 \rangle = \int_{\Gamma\backslash\mathbb{H}} f_1(z)\overline{f_2(z)}\, y^k \frac{dx\,dy}{y^2}$ defines an inner product on $S_k(\Gamma)$

* Classical Automorphic Forms

$\mathbb{H} \cong SL_2(\mathbb{R})/SO_2(\mathbb{R})$, $g.i \leftrightarrow [g]$

$SL_2(\mathbb{R}) = NAK$: Iwasawa decomposition with hyperbolic coordinates

$N = \left\{ n(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$, $A = \left\{ t(y) = \begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix} : y > 0 \right\}$

$$K = SO_2(\mathbb{R}) = \left\{ r(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}$$

Def 5. For $f \in M_k(\Gamma)$, the function on $SL_2(\mathbb{R})$: $\varphi_f(g) := f(g.i)\, j(g,i)^{-k}$ is called the automorphic lift of f.

Lemma 2. The image of automorphic lifts of $M_k(\Gamma)$ consists precisely of $\varphi \in C^\infty(SL_2(\mathbb{R}))$ s.t.

(1) $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in \Gamma$ & $g \in SL_2(\mathbb{R})$

(2) $\varphi(g r(\theta)) = e^{ik\theta}\varphi(g)$

(3) In the hyperbolic coordinates, we have

$$\left(-iy\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-\frac{1}{2i}\frac{\partial}{\partial \theta}\right)\varphi=0$$

Rmk. (3) corresponds to Cauchy-Riemann equation "$\frac{\partial}{\partial \bar z}f=0$".

Lemma 3. The image of automorphic lifts of $S_k(\Gamma)$ is $\varphi\in C^\infty$ s.t. (1)-(3) &

(4) φ is bounded. In particular, $\int_{\Gamma\backslash SL_2(\mathbb{R})}|\varphi(g)|^2\,dg<\infty$ ← Haar measure

(5) φ is cuspidal, i.e., for $g\in SL_2(\mathbb{R})$ & each cusp s of Γ

$$\int_0^1\varphi\left(\sigma_s\begin{pmatrix}1&x h\\&1\end{pmatrix}g\right)dx=0$$

where $\sigma_s.\infty=s$ & $\sigma_s^{-1}\Gamma\sigma_s\cap N=\{n(hk)\mid k\in\mathbb{Z}\}$.

Moreover, we have identification of inner products

$$\int_{\Gamma\backslash H}f_1(z)\overline{f_2(z)}(\operatorname{Im}z)^k\frac{dx\,dy}{y^2}=\int_{\Gamma\backslash SL_2(\mathbb{R})}\varphi_{f_1}(g)\overline{\varphi_{f_2}(g)}\,dg$$

with $dg=dx\,\frac{dy}{y^2}\,\frac{d\theta}{2\pi}$ in the hyperbolic coordinates.

Upshot: $S_k(\Gamma)$ is realized as a subspace of $L^2(\Gamma\backslash SL_2(\mathbb{R}))$. What makes this possible essentially is "$\Gamma\backslash H\simeq\Gamma\backslash SL_2(\mathbb{R})/SO_2(\mathbb{R})$", identification of fund. domains.

Question: What happens if we let Γ runs through a direct system, such as $\Gamma=\Gamma_0(N)$ for $N\in\mathbb{Z}_{>0}$? (Anything similar to Dirichlet characters?)

* Lattices Over Dedekind Domains (see [OM73, §81])

We start with some elementary properties of Dedekind domain: $\mathcal{O}$, $F=\operatorname{Frac}(\mathcal{O})$

Prop. 1. Let $\mathfrak{a},\mathfrak{b},\mathfrak{c}$ be fractional ideals. Then $\exists\,\alpha,\beta\in F^\times$ s.t.

$$\mathfrak{c}=\alpha\mathfrak{a}+\beta\cdot\mathfrak{b}$$

In fact, any β with $\beta\cdot\mathfrak{b}\subseteq\mathfrak{c}$ can be taken above.

Proof. Choose $\beta\in\mathfrak{b}^{-1}\mathfrak{c}-\{0\}$. Let $S\subset\operatorname{Spec}(\mathcal{O})$ be finite s.t. $\forall\,\mathfrak{p}\notin S\Rightarrow$

$$\operatorname{ord}_{\mathfrak{p}}(\mathfrak{a})=\operatorname{ord}_{\mathfrak{p}}(\mathfrak{b})=\operatorname{ord}_{\mathfrak{p}}(\mathfrak{c})=\operatorname{ord}_{\mathfrak{p}}(\beta)=0$$

Choose $\alpha\in F^\times$ via strong approximation theorem s.t.

$\{\operatorname{ord}_{\mathfrak{p}}(\alpha)=\operatorname{ord}_{\mathfrak{p}}(\mathfrak{c})-\operatorname{ord}_{\mathfrak{p}}(\mathfrak{a})$ for all $\mathfrak{p}\in S$

$\quad \left\{ \mathrm{ord}_{\mathfrak{P}}(\alpha) = 0 \right.$ for $\mathfrak{P} \notin S$

Then $\mathrm{ord}_{\mathfrak{P}}(\alpha\mathfrak{a} + \beta\mathfrak{b}) = \min(\mathrm{ord}_{\mathfrak{P}}(\alpha\mathfrak{a}), \mathrm{ord}_{\mathfrak{P}}(\beta\mathfrak{b})) = \mathrm{ord}_{\mathfrak{P}}(\mathfrak{c}) \quad \forall \mathfrak{P}$

$\Rightarrow \alpha\mathfrak{a} + \beta\mathfrak{b} = \mathfrak{c}$ $\square$

Cor. 1. (1) Every fractional ideal is of the form $\alpha\mathcal{O} + \beta\mathcal{O}$ with $\alpha, \beta \in F^\times$.

In particular, every fractional ideal is a finitely generated $\mathcal{O}$-module.

(2) Given fractional ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{r}_1, \mathfrak{r}_2$, there are $\alpha, \beta \in F^\times$ s.t.

$$\begin{cases} \alpha\mathfrak{a} + \beta\mathfrak{b} = \mathcal{O} \\ \alpha\mathfrak{a}\mathfrak{r}_1 + \beta\mathfrak{b}\mathfrak{r}_2 = \mathfrak{r}_1 + \mathfrak{r}_2 \end{cases}$$

Proof. We only treat (2). It suffices to treat the special case $\mathfrak{r}_1 + \mathfrak{r}_2 = \mathcal{O}$

By Prop. 1. we can find $\alpha \in F^\times$ s.t. $\alpha\mathfrak{a} + \mathfrak{r}_2 = \mathcal{O}$, then $\beta \in F^\times$ s.t.

$\beta\mathfrak{b} + \alpha\mathfrak{a}\mathfrak{r}_1 = \mathcal{O}$. Then clearly $\beta\mathfrak{b} + \alpha\mathfrak{a} = \mathcal{O}$. Moreover,

$$\left.\begin{matrix} \mathfrak{r}_1 + \mathfrak{r}_2 = \mathcal{O} \\ \alpha\mathfrak{a} + \mathfrak{r}_2 = \mathcal{O} \end{matrix}\right\} \Rightarrow \left.\begin{matrix} \alpha\mathfrak{a}\mathfrak{r}_1 + \mathfrak{r}_2 = \mathcal{O} \\ \alpha\mathfrak{a}\mathfrak{r}_1 + \beta\mathfrak{b} = \mathcal{O} \end{matrix}\right\} \Rightarrow \alpha\mathfrak{a}\mathfrak{r}_1 + \beta\mathfrak{b}\mathfrak{r}_2 = \mathcal{O}. \quad \square$$

Fractional ideals are "lattices of rank 1".

Let V be a finite dim^l vector space $/F$.

Def. 6. An $\mathcal{O}$-submodule $M \subseteq V$ is a lattice in V if $\exists$ basis $\{x_i\}_{i=1}^n$ for V

s.t. $M \subseteq \mathcal{O}x_1 + \cdots + \mathcal{O}x_n$. If moreover $FM = V$, we call M

a lattice on V.

Rmk: The above condition is equivalent to finite generation but more convenient in the sequel.

Lemma 4. Let L be a lattice on V. Then an $\mathcal{O}$-module $M \subseteq V$ is a lattice

in V iff $\exists\, \alpha \in \mathcal{O} - \{0\}$ s.t. $\alpha M \subseteq L$.

Assume L is a lattice on V.

Thm. 1. Let $\{x_1, \cdots, x_n\}$ be a basis of V. Then $\exists$ a basis $\{y_1, \cdots, y_n\}$ with

$y_i \in Fx_1 + \cdots + Fx_i$, fractional ideals $\mathfrak{a}_1, \cdots, \mathfrak{a}_n$, s.t.

$L = \mathfrak{a}_1 y_1 + \cdots + \mathfrak{a}_n y_n$

Thm. 2. There is a fractional ideal $\mathfrak{a}$ & a basis $\{z_i\}_{i=1}^n$ for V s.t.

$$L = \mathfrak{a} z_1 + \mathcal{O} z_2 + \cdots + \mathcal{O} z_n .$$

Moreover, $\mathfrak{a}$ is determined by L up to a scalar in $F^\times$.

Assume L & K are lattices on V.

Thm. 3. There is a basis $\{x_i\}_{i=1}^n$ for V s.t.

$$\begin{cases} L = \mathfrak{a}_1 x_1 + \cdots + \mathfrak{a}_n x_n \\ K = \mathfrak{a}_1 \mathfrak{r}_1 x_1 + \cdots + \mathfrak{a}_n \mathfrak{r}_n x_n \end{cases}$$

where $\mathfrak{a}_i$ & $\mathfrak{r}_i$ are fractional ideals with $\mathfrak{r}_1 \supseteq \cdots \supseteq \mathfrak{r}_n$.

The $\mathfrak{r}_i$ determined in this way are unique, called invariant factors.

The following strengthening of Thm. 3. seems to be new.

Thm. 4. In Thm. 3. we can assume $\mathfrak{a}_2 = \mathfrak{a}_3 = \cdots = \mathfrak{a}_n = \mathcal{O}$ if $n \geq 2$.

Proof. The case $n \geq 3$ follows by success applications of the case $n=2$.

Let $n=2$. By Cor. 1. we can find $\alpha_1, \alpha_2 \in F^\times$ s.t. (note $\mathfrak{r}_1 \supseteq \mathfrak{r}_2 \Rightarrow \mathfrak{r}_1^{-1} \subseteq \mathfrak{r}_2^{-1}$)

$$\begin{cases} \alpha_1 \mathfrak{a}_1^{-1} + \alpha_2 \mathfrak{a}_2^{-1} = \mathcal{O} & (\Rightarrow \alpha_1 \in \mathfrak{a}_1, \alpha_2 \in \mathfrak{a}_2) \\ \alpha_1 \mathfrak{a}_1^{-1} \mathfrak{r}_1^{-1} + \alpha_2 \mathfrak{a}_2^{-1} \mathfrak{r}_2^{-1} = \mathfrak{r}_2^{-1} & \Leftrightarrow \alpha_1 \mathfrak{a}_1^{-1} \mathfrak{r}_2 \mathfrak{r}_1^{-1} + \alpha_2 \mathfrak{a}_2^{-1} = \mathcal{O} \end{cases}$$

We can thus find $\beta_1 \in \mathfrak{a}_1^{-1} \mathfrak{r}_2 \mathfrak{r}_1^{-1}$, $\beta_2 \in \mathfrak{a}_2^{-1}$ s.t. $\alpha_1 \beta_1 - \alpha_2 \beta_2 = 1$

Consider the change of base

$$\begin{cases} y_1 = \alpha_1 x_1 + \alpha_2 x_2 \\ y_2 = \beta_2 x_1 + \beta_1 x_2 \end{cases} \Leftrightarrow \begin{cases} x_1 = \beta_1 y_1 - \alpha_2 y_2 \\ x_2 = -\beta_2 y_1 + \alpha_1 y_2 \end{cases}$$

and the lattices $\quad L' = \mathcal{O} y_1 + \mathfrak{a}_1 \mathfrak{a}_2 y_2 \quad K' = \mathfrak{r}_2 y_1 + \mathfrak{a}_1 \mathfrak{a}_2 \mathfrak{r}_1 y_2$

We verify:
- $L' \subseteq L$ by $\alpha_1 \in \mathfrak{a}_1$, $\alpha_2 \in \mathfrak{a}_2$, $\beta_2 \mathfrak{a}_1 \mathfrak{a}_2 \subseteq \mathfrak{a}_1$, $\beta_1 \mathfrak{a}_1 \mathfrak{a}_2 \subseteq \mathfrak{a}_2$
- $L \subseteq L'$ by $\beta_1 \mathfrak{a}_1 \subseteq \mathcal{O}$, $\alpha_2 \mathfrak{a}_1 \subseteq \mathfrak{a}_1 \mathfrak{a}_2$, $\beta_2 \mathfrak{a}_2 \subseteq \mathcal{O}$, $\alpha_1 \mathfrak{a}_2 \subseteq \mathfrak{a}_1 \mathfrak{a}_2$
- $K' \subseteq K$ by $\alpha_1 \mathfrak{r}_2 \subseteq \mathfrak{a}_1 \mathfrak{r}_1$, $\alpha_2 \mathfrak{r}_2 \subseteq \mathfrak{a}_2 \mathfrak{r}_2$, $\beta_2 \mathfrak{a}_1 \mathfrak{a}_2 \mathfrak{r}_1 \subseteq \mathfrak{a}_1 \mathfrak{r}_1$, $\beta_1 \mathfrak{a}_1 \mathfrak{a}_2 \mathfrak{r}_1 \subseteq \mathfrak{a}_2 \mathfrak{r}_2$
- $K \subseteq K'$ by $\beta_1 \mathfrak{a}_1 \mathfrak{r}_1 \subseteq \mathfrak{r}_2$, $\alpha_2 \mathfrak{a}_1 \mathfrak{r}_1 \subseteq \mathfrak{a}_1 \mathfrak{a}_2 \mathfrak{r}_1$, $\beta_2 \mathfrak{a}_2 \mathfrak{r}_2 \subseteq \mathfrak{r}_2$, $\alpha_1 \mathfrak{a}_2 \mathfrak{r}_2 \subseteq \mathfrak{a}_1 \mathfrak{a}_2 \mathfrak{r}_1$

Hence $L = L'$ & $K = K'$ and we are done with the case $n=2$. $\square$

* Local-Global Principle for Lattices

Choose a basis $\{e_i\}_{i=1}^n$ of V s.t. $V = Fe_1 + \cdots + Fe_n \simeq F^n$. Take $L^0 = \mathcal{O}e_1 + \cdots + \mathcal{O}e_n$

For any $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$, let $V_\mathfrak{p} = F_\mathfrak{p} \otimes_F V = F_\mathfrak{p} e_1 + \cdots + F_\mathfrak{p} e_n \supseteq L_\mathfrak{p}^0 = \mathcal{O}_\mathfrak{p} \otimes_\mathcal{O} L^0$

Lemma 5. For any lattice L on V we have $L_\mathfrak{p} := \mathcal{O}_\mathfrak{p} \otimes_\mathcal{O} L = L_\mathfrak{p}^0$

for a.e. $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$.

Proof. By Thm. 4. we can find a basis $\{x_i\}$ of V s.t.

$$\begin{cases} L^0 = \mathfrak{a}x_1 + \mathcal{O}x_2 + \cdots + \mathcal{O}x_n \\ L = \mathfrak{a}\mathfrak{r}_1 x_1 + \mathfrak{r}_2 x_2 + \cdots + \mathfrak{r}_n x_n \end{cases} \quad \text{for fractional ideals } \mathfrak{a}, \mathfrak{r}_i.$$

For $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$ not appearing in the factorizations of $\mathfrak{a}, \mathfrak{r}_i$

we have $\mathfrak{a}_\mathfrak{p} = (\mathfrak{r}_i)_\mathfrak{p} = \mathcal{O}_\mathfrak{p}$.

Hence $L_\mathfrak{p}^0 = L_\mathfrak{p} = \mathcal{O}_\mathfrak{p} x_1 + \cdots + \mathcal{O}_\mathfrak{p} x_n$ for such $\mathfrak{p}$. $\square$

Def. 7. Let $\mathcal{L}$ (resp. $\mathcal{L}_\mathfrak{p}$) be the set of lattices in V (resp. $V_\mathfrak{p}$).

Form the restricted product $\prod_\mathfrak{p}' \mathcal{L}_\mathfrak{p}$ w.r.t. $L_\mathfrak{p}^0$.

Exercise: Show that $\prod_\mathfrak{p}' \mathcal{L}_\mathfrak{p}$ is independent of the choice of $\{e_i\}_{i=1}^n$.

For any subset $S \subseteq V$ we write $\overline{S}_\mathfrak{p}$ for the closure of S in $V_\mathfrak{p}$.

Prop. 2. The map $\iota: \mathcal{L} \to \prod_\mathfrak{p}' \mathcal{L}_\mathfrak{p}$, $L \mapsto (\overline{L}_\mathfrak{p})_\mathfrak{p}$ is bijective.

Rmk: The case $n=1$ is equivalent to the unique prime factorization of fractional ideals.

Proof. It suffices to verify that the inverse of ι is given by

$$\tau: \prod_\mathfrak{p}' \mathcal{L}_\mathfrak{p} \to \mathcal{L}, \quad (L_\mathfrak{p})_\mathfrak{p} \mapsto \bigcap_\mathfrak{p} (L_\mathfrak{p} \cap V)$$

(1) τ is well-defined.

Let $S \subset \mathrm{Spec}(\mathcal{O})$ be a finite set s.t. $\mathfrak{p} \notin S \Rightarrow L_\mathfrak{p} = L_\mathfrak{p}^0$. For $\mathfrak{p} \in S$

let $n_\mathfrak{p}, m_\mathfrak{p} \in \mathbb{Z}$ s.t. $\mathfrak{p}^{n_\mathfrak{p}} L_\mathfrak{p}^0 \subset L_\mathfrak{p} \subset \mathfrak{p}^{m_\mathfrak{p}} L_\mathfrak{p}^0$

$$\Rightarrow \left(\prod_{\mathfrak{p} \in S} \mathfrak{p}^{n_\mathfrak{p}}\right) L^0 \subset \bigcap_\mathfrak{p} (L_\mathfrak{p} \cap V) \subset \left(\prod_{\mathfrak{p} \in S} \mathfrak{p}^{m_\mathfrak{p}}\right) L^0$$

Hence $\mathrm{Im}(\tau) \subseteq \mathcal{L}$.

(2) $\tau \circ \iota = \mathrm{id}$.

for $L \in \mathcal{L}$ find basis $\{x_i\}$ & fractional ideals $\mathfrak{a}_i$ s.t. $L = \mathfrak{a}_1 x_1 + \cdots + \mathfrak{a}_n x_n$

Then $\mathcal{I}_\mathfrak{p} = \mathfrak{a}_{1,\mathfrak{p}} x_1 + \cdots + \mathfrak{a}_{n,\mathfrak{p}} x_n$ & $\bigcap_\mathfrak{p} \mathcal{I}_\mathfrak{p} = \mathfrak{a}_1 x_1 + \cdots + \mathfrak{a}_n x_n$

since $\bigcap_\mathfrak{p} (\mathfrak{a}_{i,\mathfrak{p}} \cap F) = \mathfrak{a}_i$ by the case $n=1$.

(3) $\iota \circ \tau = id$

Let $(L_\mathfrak{p})_\mathfrak{p} \in \prod'_\mathfrak{p} \mathcal{L}_\mathfrak{p}$ & $L = \bigcap_\mathfrak{p} (L_\mathfrak{p} \cap V)$. Let $S \subset \operatorname{Spec}(\mathcal{O})$ be finite

s.t. $\mathfrak{p} \notin S \Rightarrow \mathcal{I}_\mathfrak{p} = L_\mathfrak{p} = L^\circ_\mathfrak{p}$.

For any $v_\mathfrak{p} \in L_\mathfrak{p}$, $\mathfrak{p} \in S$, we can find $v \in V$ s.t. v is arbitrarily close to $v_\mathfrak{p}$ in $V_\mathfrak{p}$ at $\mathfrak{p} \in S$ & $v \in L^\circ_\mathfrak{p}$ at $\mathfrak{p} \notin S$, by Strong Approx. Thm.

In particular, we can assume $v \in L_\mathfrak{p}$ at $\mathfrak{p} \in S$, since $L_\mathfrak{p}$ is open.

Thus $v \in L$ & arbitrarily close to $L_\mathfrak{p}$ at $\mathfrak{p} \in S \Rightarrow \mathcal{I}_\mathfrak{p} = L_\mathfrak{p}$ at $\mathfrak{p} \in S$.

Hence $\mathcal{I}_\mathfrak{p} = L_\mathfrak{p}$ at all $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}) \Rightarrow \iota \circ \tau((L_\mathfrak{p})_\mathfrak{p}) = (L_\mathfrak{p})_\mathfrak{p}$. $\square$

* Strong Approximation via Lattices

Viewing vectors in $V \simeq F^n$ (resp. $V_\mathfrak{p} \simeq F_\mathfrak{p}^n$) as row vectors, we have a right action of $GL_n(F)$ (resp. $GL_n(F_\mathfrak{p})$) on V (resp. $V_\mathfrak{p}$), hence on $\mathcal{L}$ & $\mathcal{L} \times \mathcal{L}$ (resp. $\mathcal{L}_\mathfrak{p}$ & $\mathcal{L}_\mathfrak{p} \times \mathcal{L}_\mathfrak{p}$).

Lemma 6. (1) The action of $GL_n(F)$ on $\mathcal{L} \times \mathcal{L}$ preserves the invariant factors.

(2) Fixing invariant factors $\underline{r} := (r_1, \cdots, r_n)$ with $r_1 \geq \cdots \geq r_n$ & define

$$X_{\underline{r}} := \{ (L, K) \in \mathcal{L} \times \mathcal{L} \mid \text{invariant factors of } (L,K) \text{ as in Thm.3} = \underline{r} \}$$

Then the orbits of $GL_n(F)$ in $X_{\underline{r}}$ are in bijection with the class group $Cl(\mathcal{O})$ ($= Cl(F)$) of $\mathcal{O}$.

Proof. (1) is obvious. To prove (2) we use Thm.4. & Thm.2.:

Two pairs of lattices in $X_{\underline{r}}$

$$\begin{cases} L = \mathfrak{a} x_1 + \mathcal{O} x_2 + \cdots + \mathcal{O} x_n \\ K = \mathfrak{a} r_1 x_1 + r_2 x_2 + \cdots + r_n x_n \end{cases} \qquad \begin{cases} L' = \mathfrak{a}' y_1 + \mathcal{O} y_2 + \cdots + \mathcal{O} y_n \\ K' = \mathfrak{a}' r_1 y_1 + r_2 y_2 + \cdots + r_n y_n \end{cases}$$

are in the same orbit of $GL_n(F)$ iff. $\mathfrak{a} \sim \mathfrak{a}'$ in $Cl(\mathcal{O})$.

Rmk. We have similar notation/statements for $\underline{r}_\mathfrak{p}$, $X_{\underline{r}_\mathfrak{p}}$, $GL_n(F_\mathfrak{p})$ etc.

We also have $X_{\underline{r}} \simeq \prod_\mathfrak{p}' X_{\underline{r}_\mathfrak{p}}$ by Prop. 2.

Thm. 5. For any invariant factors $\underline{r} = (r_1, \cdots, r_n)$, let $K[\underline{r}] < GL_n(\mathbb{A}_{fin})$ be the stabilizer group of the pair of lattices (in $\prod_\mathfrak{p}' X_{\underline{r}_\mathfrak{p}}$)

$$\begin{cases} L^0 = \mathcal{O}e_1 + \cdots + \mathcal{O}e_n \\ K^0 = r_1 e_1 + \cdots + r_n e_n \end{cases}$$

Then the determinant map induces a bijection

$$K[\underline{r}] \backslash GL_n(\mathbb{A}_{fin}) / GL_n(F) \simeq \hat{\mathcal{O}}^\times \backslash \mathbb{A}_{fin}^\times / F^\times = Cl(\mathcal{O})$$

Proof. By Lemma 6, $GL_n(\mathbb{A}_{fin})$ acts on $X_{\underline{r}}$ transitively with $K[\underline{r}]$ being the stabilizer group. Hence $K[\underline{r}] \backslash GL_n(\mathbb{A}_{fin}) \simeq X_{\underline{r}}$ on which $GL_n(F)$-orbits are in bijection with $Cl(\mathcal{O})$. The desired equality follows by chasing these bijections.

* Adelic Automorphic Lifts

Define $\Gamma_0(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}) \mid N \mid c \}$ &

$$K_0[N] = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{A}_{fin}) \mid N \mid c_\mathfrak{p}\ \forall \mathfrak{p} \}$$

Then Thm. 5. implies

$$GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}) / K_0[N] \simeq \Gamma_0(N) \backslash GL_2(\mathbb{R}).$$

Regarding $\mathbb{R}_{>0}$ as a diagonal central subgp. of $GL_2(\mathbb{R}) < GL_2(\mathbb{A})$, we get

$$GL_2(\mathbb{Q})\mathbb{R}_{>0} \backslash GL_2(\mathbb{A}) / K_0[N] \simeq \Gamma_0(N)\mathbb{R}_{>0} \backslash GL_2(\mathbb{R}) \simeq \Gamma_0(N) \backslash SL_2(\mathbb{R})$$

which is another form of the fundamental domain for classical automorphic for for $\Gamma_0(N)$. We can thus identify classical automorphic forms as functions on $GL_2(\mathbb{A})$. We can do this for $M_k(\Gamma_0(N), \chi)$ with Dirichlet characters χ.

Prop. 3. For Dirichlet character χ_1, let $\chi : \mathbb{Q}^\times \mathbb{R}_{>0} \backslash \mathbb{A}^\times \to \mathbb{C}^1$ be the Hecke character associated to χ_1 by the final Remark of §1.1. The $M_k(\Gamma_0(N), \chi_1)$ is in bijection with smooth functions φ on $GL_2(\mathbb{A})$ s.t. $(\chi_1(-1) = (-1)^k)$

(1) $\varphi(z\gamma g x) = \chi^{-1}(z)\,\varphi(g)$ for all $z \in \mathbb{A}^\times$, $\gamma \in GL_2(\mathbb{Q})$,

$x \in K_1[N] = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0[N] \mid d \equiv 1 \ (N\hat{\mathbb{Z}})\}$ & $g \in GL_2(\mathbb{A})$

(2) $\varphi((g_\infty r(\theta), g_{fin})) = e^{ik\theta}\varphi((g_\infty, g_{fin}))$

(3) $E_- . \varphi(\cdot, g_{fin}) = 0$ with E_- in hyperbolic coordinates given by the formula in Lemma 2. (3)

For $S_k(\Gamma(N), \chi_1)$ we put additional conditions

(4) φ is bounded, in particular

$$\int_{GL_2(\mathbb{Q})\mathbb{A}^\times \backslash GL_2(\mathbb{A})} |\varphi(g)|^2 \, dg < \infty$$

(5) φ is cuspidal, i.e., for any $g \in GL_2(\mathbb{A})$

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \varphi(n(x)g) = 0$$

Rmk: We can identify Petersson inner product if Haar measures are properly chosen.

§2.2 Spectral Decomposition v.s. Fourier inversion

* Generalities.

The adelization of M.F.s leads to

Def.8. For $\omega: F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^1$, called central character, define $L^2(GL_2, \omega)$ consisting of measurable $\varphi: GL_2(\mathbb{A}) \to \mathbb{C}$ s.t.

(1) $\varphi(\gamma z g) = \omega(z)\varphi(g)$ for all $\gamma \in GL_2(F)$, $z \in \mathbb{A}^\times$, $g \in GL_2(\mathbb{A})$

(2) $\int_{GL_2(F)\mathbb{A}^\times \backslash GL_2(\mathbb{A})} |\varphi(g)|^2 \, dg < \infty$

Define $L_0^2(GL_2, \omega)$ be the subspace $\varphi \in L^2(GL_2, \omega)$ with

(3) $\varphi_N(g) := \int_{F\backslash\mathbb{A}} \varphi(n(x)g)\, dx = 0$ a.e. $g \in GL_2(\mathbb{A})$

Rmk: $L^2(GL_2, \omega)$ inherits the right regular action (denoted by R) of $GL_2(\mathbb{A})$. It is the analogue of $L^2(F^\times\backslash\mathbb{A}^\times)$ in the setting of GL_1. Hence the unitary irreducible reps. π play the rôle of Hecke characters $\chi: F^\times\backslash\mathbb{A}^\times \to \mathbb{C}^1$.

More generally, they fit into the following frameworks together with the classical Fourier analysis on $L^2(\mathbb{R})$ & $L^2(\mathbb{Z}\backslash\mathbb{R})$.

Framework 1: Spectral decomposition

G: locally compact (unimodular) group

$\widehat{G}$: unitary dual with Fell topology

(R, V_R): unitary representation of G in V_R Hilbert space $\langle \cdot,\cdot \rangle_R$

Def. 9. The spectral decomposition of $\langle \cdot,\cdot \rangle_R$ on $V_R \times V_R$ is find & establish

(1) Borel measure $d\mu = d\mu_R$ on $\widehat{G}$, called Plancherel measure for $\langle \cdot,\cdot \rangle_R$

(2) $\forall \pi \in \mathrm{Supp}(d\mu) \subset \widehat{G}$, a G-intertwiner $F_\pi : V_R \to V_\pi$, called spectral projector, where V_π is a (possibly infinite) direct sum of some model Hilbert space for π. And V_π is called the π-isotypic component of (R, V_R).

(3) For every $v \in V_R$, $F_\pi(v)$ is well-defined for a.a. π for $d\mu$ &

$$\langle v_1, v_2 \rangle_R = \int_{\widehat{G}} \langle F_\pi(v_1), F_\pi(v_2) \rangle_\pi \, d\mu(\pi)$$ (also called "Plancherel formula")

Rmk: The above is sometimes paraphrased as $V_R \simeq \int_{\widehat{G}} V_\pi \, d\mu(\pi)$ in literature.

Def. 10. Let $V \subset V_R$ be a G-stable subspace defined with a finer topology. Let $\widetilde{V}$ ($\supset V_R$) be the topological dual of V. The spectral decomposition of the natural pairing $\langle \cdot,\cdot \rangle$ on $V \times \widetilde{V}$ is to find & establish

(4) $\forall \pi \in \mathrm{Supp}(d\mu) \subset \widehat{G}$, an extension $\widetilde{F}_\pi : \widetilde{V} \to \widetilde{V}_\pi$ of F_π to some topological dual $\widetilde{V}_\pi$ of $F_\pi(V) \subset V_\pi$ with $\langle \cdot,\cdot \rangle_\pi$ on $F_\pi(V) \times \widetilde{V}_\pi$

(5) For every $l \in \widetilde{V}$, $\widetilde{F}_\pi(l)$ is well-defined for a.a. π for $d\mu$ &

$$\langle v, l \rangle = \int_{\widehat{G}} \langle F_\pi(v), \widetilde{F}_\pi(l) \rangle_\pi \, d\mu(\pi)$$

If (4) & (5) are established for a particular $l \in \widetilde{V}$, we call the formula in (5) the spectral decomposition of l (on V).

Framework 2: Fourier inversion

X: topological manifold with right G-action

dx: G-invariant measure on X

$V_R := L^2(X, dx)$

F_π^*: adjoint of F_π s.t. $\langle F_\pi^* F_\pi(v_1), v_2\rangle_R = \langle F_\pi(v_1), F_\pi(v_2)\rangle_\pi$

Existence, domain & image of F_π^* are all subtle questions!

Rmk: If G & X have smooth structures, one can define V_R^∞, V_π^∞ etc. & suitable Dixmier-Malliavin's Thm implies $F_\pi(V_R^\infty) \subset V_\pi^\infty$.

Rmk: During the establishment of Plancherel formula, one usually has already obtained

$$F_\pi^*(V_\pi^\infty) \subset V_R^\infty \quad \& \quad v(x) = \int_{\hat{G}} (F_\pi^* F_\pi v)(x)\, d\mu(\pi) \qquad (1)$$

for $v \in V_0$ for a sufficiently large $V_0 \subset V_R^\infty$ with strong convergence in $x \in X$, and possibly also an estimation of the dominant integral

$$x \mapsto \int_{\hat{G}} |(F_\pi^* F_\pi v)(x)|\, d\mu(\pi) \qquad (2)$$

as $x \in X$ varies.

Def. 1.1. Fourier inversion is the extension of (1) to larger subspaces $V_0 \subset V \subset V_R$ together with an estimation of its dominant integral (2).

* Examples: Classical Fourier analysis

(A) $G = X = \mathbb{R}$, $\pi = \pi_\xi$ parametrized by $\xi \in \mathbb{R}$ & 1-dim'l $V_\xi = \mathbb{C} e_\xi$ & $\pi_\xi(x) e_\xi = e^{2\pi i \xi x}$

Let $F_\xi := F_{\pi_\xi}$ then the 1st step is the Fourier inversion & Plancherel formula for

$$V_0 = \{ P(x) e^{-ax^2} \mid a > 0,\ P \in \mathbb{C}[x] \}$$

s.t. $\underbrace{F_\xi(v)}_{=\hat{v}(\xi)} = \int_{\mathbb{R}} v(x) e^{-2\pi i \xi x}\, dx$ for $v \in V_0$

Extension to $V = C_c^\infty(\mathbb{R})$ or $S(\mathbb{R})$ requires

(1) $|e^{2\pi i \xi x}| \leq 1$ coarse sup norm bound (CSB)

(2) $|\hat{v}(\xi)| \ll_A (1+|\xi|)^{-A}$ & $\int_{\mathbb{R}} (1+|\xi|)^{-A} d\xi < \infty$, $\forall A > 1$

weak Weyl's law (WWL)

(B) $G = \mathbb{R}$, $X = \mathbb{Z}\backslash\mathbb{R}$ $\leadsto$ Plancherel measure $d\mu$ supported in $\mathbb{Z}$

$V_0 = C^\infty(X)$ F.I. & Pl.F. require (CSB) & (UWL):

$|\mathfrak{D}(n)| \ll_A (1+|n|)^{-A}$ & $\sum_{n\in\mathbb{Z}} (1+|n|)^{-A} < \infty \quad \forall A>1$

§2.3 Automorphic Setting: Plancherel Formula / Spectral Decomposition

* This is essentially a summary & extension of §2-4 in [GJ 79].

Def. 12. $\mathcal{P}(G_2, \omega) = \{ P(f) \mid f \in C^\infty(G_2(\mathbb{A})),\ \begin{array}{l} f(n\gamma z g) = \omega(z) f(g),\ n\in N(\mathbb{A}), \gamma\in B(F), z\in Z(\mathbb{A}) \\ Z(\mathbb{A})N(\mathbb{A})B(F)\setminus \mathrm{supp}(f) \text{ is compact} \end{array} \}$

where $B = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\} \subset G_2$ & $P(f)(g) := \sum_{\gamma\in B(F)\setminus G_2(F)} f(\gamma g)$ is called

P-series or incomplete θ-series.

Once well-defineness is verified, one sees

$$L^2(G_2, \omega) = \overline{\mathcal{P}(G_2,\omega)} \oplus L_0^2(G_2,\omega) \quad (3)$$

by the standard Rankin-Selberg unfolding, as

$$\langle \varphi, P(f)\rangle = \int_{G_2(F)\mathbb{A}^\times\backslash G_2(\mathbb{A})} \varphi(g)\, \overline{P(f)(g)}\, dg$$

$$= \int_{B(F)\mathbb{A}^\times\backslash G_2(\mathbb{A})} \varphi(g)\, \overline{f(g)}\, dg$$

$$= \int_{B(F)N(\mathbb{A})\mathbb{A}^\times\backslash G_2(\mathbb{A})} \Big(\underbrace{\int_{N(\mathbb{A})} \varphi(ng)\,dn}_{=\varphi_N(g)}\Big)\, \overline{f(g)}\, dg = 0 \quad \text{for all } P(f)$$

$\Leftrightarrow \quad \varphi_N(g) = 0$ a.e. g

From (3), [GJ 79] reduces the problem of Plancherel formula for $L^2(G_2,\omega)$ to that for $\overline{\mathcal{P}(G_2,\omega)}$ and $L_0^2(G_2,\omega)$ respectively.

* Cuspidal Part: $L_0^2(G_2,\omega)$

This part is similar to the regular representation of a compact group, hence decomposes a direct sum of unitary irreducible representations of $G_2(\mathbb{A})$. The proof is an adaptation of that of the Peter-Weyl theorem, namely based on the following:

Lemma 7. Let $0 \neq \mathcal{R}$ be a $*$-closed subalgebra of compact operators on a Hilbert space V, which contains an approximation of identity. Then V is decomposable for $\mathcal{R}$, and each irreducible irreducible representation occurs with finite multiplicity.

Lemma 8. Let X be a locally compact space with a finite positive measure μ such that $L^2(X,\mu)$ is separable. Let T be a linear map from V into the vector space of bounded continuous functions on X. Assume there exists $C>0$ s.t. for all $f\in V$ we have

$$\|Tf\|_\infty \le C\,\|f\|_2$$

where $\|\cdot\|$ is the sup norm on X. Then $T: V \to L^2(X)$ is a compact operator, which can be represented by a kernel function in $L^2(X\times X, \mu\times\mu)$.

For a proof of Lemma 7 & 8, see Chap. I of [Lan 85].

Let $f \in C_c^\infty(G_2, \omega^{-1})$ and form kernel functions

$$J_f(x,y) := \sum_{\alpha\in F} f(x^{-1}n(\alpha)y) - \int_{\mathbb{A}} f(x^{-1}n(u)y)\,du$$

$$K_f(x,y) := \sum_{\gamma\in G_2(F)} f(x^{-1}\gamma y).$$

Then for any $\varphi \in L_0^2(G_2,\omega)$ we have

$$\int_{[PG_2]} K_f(x,y)\varphi(y)\,dy = R(f)\varphi(x) = \int_{N(F)Z(\mathbb{A})\backslash G_2(\mathbb{A})} J_f(x,y)\varphi(y)\,dy.$$

The following estimation ensures the applicability of Lemma 7 & 8 to the automorphic setting:

Prop. 4. For any $f\in C_c^\infty(G_2,\omega^{-1})$, there is $C_f>0$ s.t. for all $\varphi\in L_0^2(G_2,\omega)$

$$\|R(f)\varphi\|_\infty \le C_f\,\|\varphi\|_2.$$

For a proof, see [Go 66].

Rmk: The subspace $V_{0,0} \subseteq L_0^2(G_2,\omega)$ for which one has Fourier inversion is not specified in the above proof. However, we can indeed infer that a basis for the Hilbert space $L_0^2(G_2,\omega)$ can be taken as K-finite (in fact isotypic) hence smooth functions in $C^\infty(G_2,\omega)$. Moreover, they have rapid decay in any Siegel domain by the proof in [Go 66]. By choosing f to be bi-K-finite, one should be able to infer (CSB) for such a basis by a refinement of the proof in [Go 66], although this seems to be not explicated in literature

We may take V_∞ to be the algebraic span of this basis.

* Non-cuspidal part: $\overline{\mathcal{P}(Gl_2,\omega)}$

The construction of $\mathcal{P}$-series is an $Gl_2(\mathbb{A})$-intertwining operator

$$\mathcal{P}:\ I(\omega) := \mathrm{Ind}_{N(\mathbb{A})Z(\mathbb{A})B(F)}^{Gl_2(\mathbb{A})}\, \omega \ \to\ L^2(Gl_2,\omega). \qquad (4)$$

Note that $N(\mathbb{A})Z(\mathbb{A})B(F) \lhd B(\mathbb{A}) < Gl_2(\mathbb{A})$, and by the "induction by step" we have

$$I(\omega) = \int_{\widehat{F^\times\backslash\mathbb{A}^\times}} I(\chi,\omega\chi^{-1})\, d\mu_{PL}(\chi) = \bigoplus_{\chi\in \widehat{F^\times\mathbb{R}_+\backslash\mathbb{A}^\times}} \int_{\mathbb{R}} I(\chi|\cdot|_{\mathbb{A}}^{i\tau}, \omega\chi^{-1}|\cdot|_{\mathbb{A}}^{-i\tau})\,\frac{d\tau}{2\pi} \qquad (5)$$

compact abelian

with $I(\chi,\omega\chi^{-1}) = \mathrm{Ind}_{B(\mathbb{A})}^{Gl_2(\mathbb{A})}(\chi,\omega\chi^{-1})$ "unitary principal series"

$$= \left\{ f: Gl_2(\mathbb{A})\to\mathbb{C} \ \middle|\ f\left(\begin{pmatrix} t_1 & x \\ & t_2\end{pmatrix} g\right) = \chi(t_1)\,\omega\chi^{-1}(t_2)\left|\frac{t_1}{t_2}\right|_{\mathbb{A}}^{\frac12} f(g)\right\}$$

Def. B. For $f\in I(\omega)$ & $s\in\mathbb{C}$, $\chi\in\widehat{F^\times\mathbb{R}_{>0}\backslash\mathbb{A}^\times}$ we introduce

$$f[\chi|\cdot|_{\mathbb{A}}^{s}, g] := \int_{F^\times\backslash\mathbb{A}^\times} \chi^{-1}(t)\,|t|_{\mathbb{A}}^{-(s+\frac12)} f\left(\begin{pmatrix} t & \\ & 1\end{pmatrix} g\right) d^\times t \qquad (6)$$

We have $f[\chi|\cdot|_{\mathbb{A}}^{s},\cdot\,] \in I(\chi|\cdot|_{\mathbb{A}}^{s}, \omega\chi^{-1}|\cdot|_{\mathbb{A}}^{-s})$ which is a concrete realization of (5) for $s\in i\mathbb{R}$ by the Mellin inversion

$$f(g) = \sum_{\chi} \int_{\mathrm{Re}\, s = c} f[\chi\cdot|\cdot|_{\mathbb{A}}^{s}, g]\,\frac{ds}{2\pi i} \qquad (7)$$

valid for any $c\in\mathbb{R}$.

Therefore it is reasonable to expect $\overline{\mathcal{P}(Gl_2,\omega)}$ to be decomposed as "$\int_{\widehat{F^\times\backslash\mathbb{A}^\times}} I(\chi,\omega\chi^{-1})\,d\mu_{PL}(\chi)$". To this end, we need to relate the inner product $\langle \mathcal{P}(f_1), \mathcal{P}(f_2)\rangle$ with the inner products in $I(\chi,\omega\chi^{-1})$ for the decompositions of f_1 & f_2.

The content of §3&4 [CJ79] is a computation of $\langle \mathcal{P}(f_1), \mathcal{P}(f_2)\rangle$ in terms of the decomposition (7), during which process one encounters the intertwining operators for principal series

$$\mathcal{M}:\ I(\chi|\cdot|_{\mathbb{A}}^{s}, \omega\chi^{-1}|\cdot|_{\mathbb{A}}^{-s}) \to I(\omega\chi^{-1}|\cdot|_{\mathbb{A}}^{-s}, \chi|\cdot|_{\mathbb{A}}^{s})$$

$\mathcal{M}f(g) := \int_{N(\mathbb{A})} f(wng)\,dn$, $w = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ (or $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$)

which – is abs. conv. for $\Re s > 1/2$

– admits meromorphic continuation to $s \in \mathbb{C}$ with simple pole at $s = \frac{1}{2}$ if $\chi^2 = \omega$.

The final formula is (4.21) of [GJ79]:

$$\langle P(f_1), P(f_2) \rangle$$

$$= \sum_{\chi \in \widehat{F^\times \mathbb{R}_+ \backslash \mathbb{A}^\times}} \int_{\mathbb{R}} \langle f_1[\chi|\cdot|_{\mathbb{A}}^{i\tau}, \cdot] + \mathcal{M}f_1[\omega\chi^{-1}|\cdot|_{\mathbb{A}}^{-i\tau}, \cdot], f_2[\chi|\cdot|_{\mathbb{A}}^{i\tau}, \cdot] \rangle \frac{d\tau}{2\pi}$$

$$+ c \sum_{\chi^2 = \omega} \langle f_1[\chi|\cdot|_{\mathbb{A}}^{\frac{1}{2}}, \cdot], \chi \circ \det \rangle \langle \chi \circ \det, f_2[\chi|\cdot|_{\mathbb{A}}^{\frac{1}{2}}, \cdot] \rangle \quad (8)$$

That is the Plancherel formula / Spectral decomposition for

$$\mathcal{P}(G_2, \omega) \to \bigoplus_{\chi \in \widehat{F^\times \mathbb{R}_+ \backslash \mathbb{A}^\times}} \int_{\mathbb{R}} I(\chi|\cdot|_{\mathbb{A}}^{i\tau}, \omega\chi^{-1}|\cdot|_{\mathbb{A}}^{-i\tau}) \frac{d\tau}{4\pi} \oplus \bigoplus_{\chi^2 = \omega} \chi \circ \det \quad (9)$$

The spectral projectors are defined by

- $F_{\chi, i\tau} := F_{I(\chi|\cdot|_{\mathbb{A}}^{i\tau}, \omega\chi^{-1}|\cdot|_{\mathbb{A}}^{-i\tau})} : P(f) \mapsto f[\chi|\cdot|_{\mathbb{A}}^{i\tau}, \cdot] + \mathcal{M}f[\omega\chi^{-1}|\cdot|_{\mathbb{A}}^{-i\tau}, \cdot]$
- F_χ for $\chi^2 = \omega$: $P(f) \mapsto f[\chi|\cdot|_{\mathbb{A}}^{\frac{1}{2}}, \cdot]$

Exercise: (1) Verify the well-definedness of $F_{\chi, i\tau}$ & F_χ for $\chi^2 = \omega$.

In particular, justify $F_\chi(P(f)) = \int_{[PGL_2]} P(f)(g) \overline{\chi \circ \det(g)}\, dg$ for $\chi^2 = \omega$

(2) Justify the change of constants $\left(\frac{d\tau}{2\pi} \rightsquigarrow \frac{d\tau}{4\pi}\right)$ from (8) to (9).

Rmk: The subspace $V_{0,c} \subset \overline{\mathcal{P}(G_2, \omega)}$ for which one has Fourier inversion can be taken as $\mathcal{P}(G_2, \omega)$. This is the content of Prop. 2.6. of [Wu17]. We finally take $V_0 = V_{0,0} \oplus V_{0,c}$.

Rmk: To prove the F.I. for $V_{0,c}$ one encounters the construction of Eisenstein series (as concrete "$F^*_{\chi, i\tau}$")

$$E: I(\chi|\cdot|_{\mathbb{A}}^{s}, \omega\chi^{-1}|\cdot|_{\mathbb{A}}^{-s})^\infty \to C^\infty(G_2, \omega), \quad E(f)(g) = \sum_{\gamma \in B(F)\backslash G_2(F)} f(\gamma g)$$

which is abs. conv. for $\Re s > \frac{1}{2}$, admits meromorphic cont. etc.

§2.4 Automorphic Fourier Inversion

* Alternative Approach

The extension of validity of Fourier inversion in the classical setting, say for $L^2(\mathbb{R})$, makes use of the properties of the heat kernel ($e^{-a(x-y)^2}$). The analogue in the automorphic setting is difficult to handle.
On the other hand, most applications do not require results as deep as the classical Fourier inversion (such as for one-side continuous func.).
Hence an alternative approach, that is easy to generalize to the automorphic setting, would be plausible. This is the main content of [Wu17].

Take $L^2(\mathbb{R})$ for example: Recall $V_0 = \{ P(x)e^{-ax^2} \mid a>0,\ P \in \mathbb{C}[x] \}$

(1) Fourier coefficients for $h \in C_c^\infty(\mathbb{R})$.

Denote the Fourier coefficient of h by extension as $\widetilde{h}(\xi)$. We need to justify it's equal to $\hat{h}(\xi) := \int_{\mathbb{R}} h(x) e^{-2\pi i \xi x}\,dx$

For $v \in V_0$ we have F.I. $v(x) = \int_{\mathbb{R}} \hat{v}(\xi) e^{2\pi i \xi x}\,d\xi$

$$\Rightarrow \int_{\mathbb{R}} h(x)\overline{v(x)}\,dx = \int_{\mathbb{R}}\int_{\mathbb{R}} h(x)\overline{\hat{v}(\xi)}\,e^{-2\pi i \xi x}\,d\xi\,dx = \int_{\mathbb{R}} \hat{h}(\xi)\overline{\hat{v}(\xi)}\,d\xi$$

$$\overset{\text{Pl. F.}}{\Vert} \qquad \int_{\mathbb{R}} \widetilde{h}(\xi)\overline{\hat{v}(\xi)}\,d\xi$$

By integration by parts, we have $|\hat{h}(\xi)| \ll_n (1+|\xi|)^{-n}$ $\forall n \in \mathbb{Z}_{\geq 0}$

$\Rightarrow \hat{h} \in L^2(\mathbb{R})$. Thus $\hat{h} - \widetilde{h} \perp \hat{v}$ for all $v \in V_0 \Rightarrow \hat{h} = \widetilde{h}$ in $L^2(\mathbb{R})$

$\Rightarrow \hat{h} = \widetilde{h}$ a.e.

Def. 14. We say $f \in C^\infty(\mathbb{R})$ "nice" if for any $n \in \mathbb{Z}_{\geq 0}$ we can find $\varepsilon > 0$ small s.t. $|f^{(n)}(x)| \ll_{n,\varepsilon} (1+|x|)^{-\frac{1}{2}-\varepsilon}$ (as $|x| \to +\infty$).

Exercise: Verify $\widetilde{f} = \hat{f}$ for nice $f \in C^\infty(\mathbb{R})$.

(2) Extension to nice functions.

Suppose $f \in C^\infty(\mathbb{R})$ nice.

In particular $f^{(n)} \in L^2(\mathbb{R})$ for any $n \in \mathbb{Z}_{\geq 0}$.

By integration by parts, $\widehat{f^{(n)}}(\xi) = (2\pi i \xi)^n \hat{f}(\xi) \in L^2(\mathbb{R})$

$\Rightarrow$ $E(f)(x) := \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$ is bounded as

$$|E(f)(x)| \leq \left(\int_{\mathbb{R}} (1+(2\pi\xi)^2) \, |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} (1+(2\pi\xi)^2)^{-1} d\xi \right)^{\frac{1}{2}} < \infty$$

$\Rightarrow$ $E(f) \in C^\infty(\mathbb{R})$.

Take $h \in C_c^\infty(\mathbb{R})$ we apply Fubini & Plancherel to get

$$\int_{\mathbb{R}} E(f)(x) \overline{h(x)} dx = \int_{\mathbb{R}} \hat{f}(\xi) \overline{\hat{h}(\xi)} d\xi = \int_{\mathbb{R}} f(x) \overline{h(x)} dx$$

Hence $f - E(f) \in C^\infty(\mathbb{R})$ is equal to 0 in the sence of distributions

$\Rightarrow$ $f = E(f)$ as smooth functions. This is F.I. for nice functions.

* The generalization of the above alternative approach in the automorphic setting is obvious if we assume a (WL2) for the cuspidal part.

Def. 5. Let $\varphi \in C^\infty(G_2, \omega)$. If for any X in the universal enveloping algebra of the Lie algebra of $GL_2(\mathbb{A}_\infty)$, we have $|R(X).\varphi(g)| \ll_\varepsilon Ht(g)^{\frac{1}{2}-\varepsilon}$ for any $\varepsilon > 0$ sufficiently small & g in a/any Siegel domain, then we say φ is nice.

Thm. 6. For nice $\varphi \in C^\infty(G_2, \omega)$, we have F.I.

$$\varphi(g) = \sum_{\pi:\text{cuspidal}} \sum_{e \in \mathcal{B}(\pi)} \langle \varphi, e \rangle e(g)$$
$$+ \sum_{\chi \in \widehat{F^\times \mathbb{R}_+ \backslash \mathbb{A}^\times}} \sum_{f \in \mathcal{B}(\chi, \omega\chi^{-1})} \int_{-\infty}^{\infty} \langle \varphi, E(i\tau, f) \rangle E(i\tau, f)(g) \frac{d\tau}{4\pi}$$
$$+ \frac{1}{\mathrm{Vol}([PGL_2])} \sum_{\chi^2 = \omega} \int_{[PGL_2]} \varphi(g) \overline{\chi(\det g)} dg \cdot \chi(\det g)$$

with normal convergence in $[PGL_2]$. Moreover, the dominant of the cuspidal part $\sum_{\pi} \sum_{e} |\langle \varphi, e \rangle e(g)|$ converges uniformly in any Siegel domain and is of rapid decay as $Ht(g) \to +\infty$.

Rmk. In §2.6 of [Wu, W?] the (CSB) is obtained via a study of the local

Rmk: [illegible] Sobolev inequalities in the Whittaker model. Consequently we obtained a variant of Thm. 6., in which the L.H.S. is replaced by $g - g_N$ and the R.H.S. has the Whittaker-Fourier expansion of e & $E(i\tau, g)$. The resulting convergence is uniform in any Siegel domain & the dominant is rapidly decreasing as $Ht(g) \to +\infty$.

Rmk: We can not make sense of Proposition 1.4 of [CPS90]. See the discussion in §4 [Wu17].

* The cuspidal part $L_0^2(GL_2, \omega)$ is similar with $L^2(\mathbb{Z} \backslash \mathbb{R})$, but different in that we do not a priori know the irreducible components appearing in $L_0^2(GL_2, \omega)$! On the other hand, most references in the literature on Weyl's law start with the trace formula, whose validity depends on the validity of F.I. (pre-trace formula). Hence we are at risk of a circular reasoning!

A way out is another way to establish trace formulae for bi-K-finite test functions, which exploits test functions of positive type & applies the monotone convergence theorem instead of Lebesgue dominant convergence. See [DL71] also §6 in [KL06] for a special case of K-invariant functions, for details.

Rmk: More details of the above clarification can be found in §2.1-2.2 [LPW23