Theory for GL(1): Tate's Thesis

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References:

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§1.1 Dirichlet L-Functions

* $\chi : (\mathbb{Z}/d\mathbb{Z})^\times \to \mathbb{C}^\times \ (\mathbb{C}^1)$ $\quad L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}$ (abs. conv. $\Re s > 1$)
* $\Phi \in \mathcal{S}(\mathbb{R})$ $\quad M\Phi(s) := \int_0^\infty \Phi(t) t^s \frac{dt}{t}$ (abs. conv. $\Re s > 0$ & merom. cont.) Ex. 1

$$Z(s,\Phi,\chi) := M\Phi(s)\, L(s,\chi) \quad \text{for } \Re s > 1$$

$$= \sum_{k \,(\mathrm{mod}\, d)}^{*} \chi(k) \int_0^\infty \Phi(t) t^s \frac{dt}{t} \cdot \sum_{n=0}^{\infty} \frac{1}{(dn+k)^s}$$

$$= \sum_{k \,(\mathrm{mod}\, d)}^{*} \chi(k) \int_0^\infty \Big\{ \sum_{n=0}^{\infty} \Phi((dn+k)t) \Big\} t^s \frac{dt}{t}$$

Assume $\chi(-1)\Phi(-x) = \Phi(x)$. Then $\chi(d-k) \sum_{n=0}^{\infty} \Phi((dn+d-k)t) = \chi(k) \sum_{n<0} \Phi((dn+k)t)$

$$Z(s,\Phi,\chi) = \frac{1}{2} \sum_{k \,(\mathrm{mod}\, d)}^{*} \chi(k) \int_0^\infty \Big\{ \sum_{n\in\mathbb{Z}} \Phi((dn+k)t) \Big\} t^s \frac{dt}{t}$$

Lemma 1. The Fourier transform of $x \mapsto \Phi(ax+b)$ is $e(\frac{b}{a}y)\, |a|^{-1} \hat{\Phi}(\frac{y}{a})$

for $\hat{\Phi}(y) := \int_{\mathbb{R}} \Phi(x) e(-xy)\, dx$.

Cor 1. $\sum_{n\in\mathbb{Z}} \Phi((dn+k)t) = \frac{1}{|dt|} \sum_{n\in\mathbb{Z}} e(\frac{k}{d}n) \hat{\Phi}(\frac{n}{dt})$

$$Z(s,\Phi,\chi) = \sum_{k \,(\mathrm{mod}\, d)}^{*} \chi(k) \Big\{ \int_{1/\sqrt{d}}^\infty \Big[\sum_{n\in\mathbb{Z}} \Phi((dn+k)t) \Big] t^s \frac{dt}{t} + \int_0^{1/\sqrt{d}} \frac{1}{dt} \Big[\sum_{n\in\mathbb{Z}} e(\tfrac{kn}{d}) \hat{\Phi}(\tfrac{n}{dt}) \Big] t^s \frac{dt}{t} \Big\}$$

$t \mapsto \frac{1}{t}$

$$= \sum_{k}^{*} \chi(k) \int_{1/\sqrt{d}}^\infty \Big\{ \sum_{n\in\mathbb{Z}} \Phi((dn+k)t) \Big\} t^s \frac{dt}{t} +$$

$$\frac{1}{d} \sum_{k}^{*} \sum_{\ell \,(\mathrm{mod}\, d)} \chi(k) e(\tfrac{k\ell}{d}) \int_{\sqrt{d}}^\infty \Big\{ \sum_{n\in\mathbb{Z}} \hat{\Phi}((n+\tfrac{\ell}{d})t) \Big\} \cdot t^{1-s} \frac{dt}{t}$$

$$= \sum_{k}^{*} \chi(k) \int_{1/\sqrt{d}}^\infty \Big\{ \sum_{n\in\mathbb{Z}} \Phi((dn+k)t) \Big\} t^s \frac{dt}{t} +$$

$$d^{\frac{1}{2}-s} \cdot \frac{1}{\sqrt{d}} \sum_{\ell \,(\mathrm{mod}\, d)} \Big(\sum_{k}^{*} \chi(k) e(\tfrac{k\ell}{d}) \Big) \int_{1/\sqrt{d}}^\infty \Big\{ \sum_{n\in\mathbb{Z}} \hat{\Phi}((dn+\ell)t) \Big\} \cdot t^{1-s} \frac{dt}{t}$$

Lemma 2. Assume χ is primitive (i.e. not factor through $(\mathbb{Z}/d\mathbb{Z})^\times \to (\mathbb{Z}/d^*\mathbb{Z})^\times$ for $d^* | d$).

(1) $\sum_{k}^{*} \chi(k) e(\frac{k\ell}{d}) \neq 0$ only if $\gcd(\ell,d)=1$, in which case

$$\sum_{k}^{*} \chi(k) e(\tfrac{k\ell}{d}) = \chi(\ell)^{-1} \sum_{k}^{*} \chi(k) e(\tfrac{k}{d}) =: \chi(\ell)^{-1} \cdot \tau_\chi$$

(2) $|\tau_\chi| = \sqrt{d}$. Notation: $\varepsilon_\chi = \frac{\tau_\chi}{\sqrt{d}} \Rightarrow \varepsilon_\chi \cdot \varepsilon_{\chi^{-1}} = \chi(-1)$

We obtain $Z(s,\Phi,\chi) = \sum_k^* \chi(k)\cdot \int_{1/\sqrt{d}}^{\infty} \left\{ \sum_{n\in\mathbb{Z}} \Phi((dn+k)t) \right\} t^s \frac{dt}{t}$

$$+ d^{\frac{1}{2}-s}\cdot \varepsilon_\chi \cdot \sum_\ell^* \chi^{-1}(\ell)\cdot \int_{1/\sqrt{d}}^{\infty} \left\{ \sum_{n\in\mathbb{Z}} \hat{\Phi}((dn+\ell)t) \right\} t^{1-s} \frac{dt}{t}$$

The RHS is abs. conv. for all $s\in\mathbb{C}$ & has symmetry (F.E.)

$$Z(1-s,\hat{\Phi},\chi^{-1}) = \varepsilon_\chi^{-1}\cdot d^{s-\frac{1}{2}}\cdot Z(s,\Phi,\chi)$$

Thm 1. Let $a\in\{0,1\}$ s.t. $\chi(-1)=(-1)^a$ & $\Lambda(s,\chi) = \left(\frac{d}{\pi}\right)^{\frac{s+a}{2}}\Gamma\left(\frac{s+a}{2}\right)L(s,\chi)$

Then $L(s,\chi)$ admits merom. cont. & $\Lambda(s,\chi) = \frac{1}{i^a}\varepsilon_\chi \Lambda(1-s,\chi^{-1})$.

Exercise: Treat $\zeta(s)$ similarly.

Remarks: (1) Basic tool: Poisson summation $\rightsquigarrow$ capture asymp. behavior at $t\to 0^+$

$\qquad\qquad$ Fourier analysis on $d\mathbb{Z}\backslash\mathbb{R}$

Morally, $\bigcup_d C^\infty(d\mathbb{Z}\backslash\mathbb{R})/\text{Relations} \approx \varinjlim_{d\in\mathbb{Z}_{>0}} C^\infty(d\mathbb{Z}\backslash\mathbb{R})$

$\approx$ Smooth functions on $\varprojlim_{d\in\mathbb{Z}_{>0}} d\mathbb{Z}\backslash\mathbb{R} \overset{\text{Strong Approx.}}{=} \mathbb{Q}\backslash\mathbb{A} \cong M_1(\mathbb{Q})\backslash M_1(\mathbb{A})$

Exercise: Concretely, let $\Phi_f((x_p)_{p<\infty}) = \prod_{p<\infty} \mathbb{1}_{k+d\mathbb{Z}_p}(x_p)$. Verify

$$\sum_{\alpha\in\mathbb{Q}} \Phi\otimes\Phi_f((x,0,\cdots)+(\alpha,\cdots)) = \sum_{n\in\mathbb{Z}} \Phi(x+dn+k).$$

(2) $\chi_1 : (\mathbb{Z}/d\mathbb{Z})^\times \to \mathbb{C}^\times \qquad L(s,\chi_2) = \sum_{n=1}^{\infty} \frac{\chi_2(n)}{n^s} \qquad n = \mathrm{Nr}((n))$

$$\chi_2(n) = \begin{cases} 0 & \text{if } (n,d)>1 \\ \chi_1(n \bmod d) & \text{if } (n,d)=1 \end{cases}$$

Can imagine: # field F, $\mathcal{O}_F$, $\mathfrak{m}\subseteq\mathcal{O}_F$ integral ideal, $\mathcal{J}^{\mathfrak{m}}$ = frac. ideals coprime with $\mathfrak{m}$

$\chi_1 : (\mathcal{O}_F/\mathfrak{m})^\times \to \mathbb{C}^\times \qquad \chi_2 : \mathcal{J}^{\mathfrak{m}} \to \mathbb{C}^\times$ & connection of χ_1,χ_2

$\rightsquigarrow$ Größencharakter [Ne 99, § VII.6]

Precisely, triplet $(\chi_1,\chi_2,\chi_\infty)$ with χ_1,χ_2 as above & $\chi_\infty : \prod_{v\mid\infty} F_v^\times \to \mathbb{C}^1$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cong GL_1(\mathbb{A}_\infty)$

s.t. $\chi_2((a)) = \chi_1(a)\chi_\infty(a)$ for any $a\in\mathcal{O}_F$ s.t. $\gcd((a),\mathfrak{m})=1$

Necessarily $\begin{cases} \chi_\infty|_{\mathcal{O}_F^{\mathfrak{m}}} = \mathbb{1} \text{ for } \mathcal{O}_F^{\mathfrak{m}} := \{\varepsilon\in\mathcal{O}_F^\times \mid \varepsilon\equiv 1 \pmod{\mathfrak{m}}\} \\ \chi_1(\varepsilon)\chi_\infty(\varepsilon)=1 \text{ for all } \varepsilon\in\mathcal{O}_F^\times \end{cases}$

Principal cong. subgps. of

"(Hilbert) Modular forms for GL_1"

Back to $\mathbb{Q}$:

$Cl(\mathbb{Q})=1 \iff \mathbb{A}_{fin}^\times = \mathbb{Q}^\times\cdot\hat{\mathbb{Z}}$. Consider $U_d = (1+d\hat{\mathbb{Z}})\cap\hat{\mathbb{Z}}^\times$

Then
$$\begin{cases} * \ \mathbb{Q}^\times\backslash\mathbb{R}^\times\times\mathbb{A}_{fin}^\times/U_d \simeq \mathbb{Z}^\times\backslash\mathbb{R}^\times\times\hat{\mathbb{Z}}^\times/U_d \simeq \mathbb{R}_{>0}\times\prod_{p<\infty}\hat{\mathbb{Z}}_p/(1+d\mathbb{Z}_p)\cap\mathbb{Z}_p^\times \\ \qquad\qquad \overset{\text{Chinese Reminder Thm}}{\simeq} \mathbb{R}_{>0}\times(\mathbb{Z}/d\mathbb{Z})^\times \\ * \ \text{Weak Approx.} \quad \mathbb{R}^\times\times\mathbb{A}_{fin}^\times = \mathbb{Q}^\times\left(\mathbb{R}^\times\times\prod_{p\nmid d}'\mathbb{Q}_p^\times\right)U_d \\ \text{Note} \quad \prod_{p\nmid d}'\mathbb{Q}_p^\times / \prod_{p\nmid d}\mathbb{Z}_p^\times = J^{(d)} \\ \text{Hence} \quad \mathbb{Q}^\times\backslash\mathbb{R}^\times\times\mathbb{A}_{fin}^\times/U_d \cong (\mathbb{Q}^\times\cap U_d)\backslash\mathbb{R}^\times\times J^{(d)} \end{cases}$$

$$\begin{tikzcd} \Rightarrow (\mathbb{Z}/d\mathbb{Z})^\times \arrow[r, "\simeq"] \arrow[dr, "\chi_1"'] & \mathbb{Q}^\times\mathbb{R}_{>0}\backslash\mathbb{A}^\times/U_d \arrow[r, "\simeq"] \arrow[d, dashed, "\chi"] & (\mathbb{Q}_{>0}\cap U_d)\backslash J^{(d)} \arrow[dl, "\chi_2"] \\ & \mathbb{C}^\times & \end{tikzcd}$$

Like in (1), $\bigcup_d\{\chi_1 : (\mathbb{Z}/d\mathbb{Z})^\times\to\mathbb{C}^\times\}/\text{Relations} \simeq \bigcup_d\{\chi : \mathbb{Q}^\times\mathbb{R}_{>0}\backslash\mathbb{A}^\times/U_d\to\mathbb{C}^\times\}/\text{Relations}$

$\simeq$ smooth characters $\chi : \underbrace{\mathbb{Q}^\times\mathbb{R}_{>0}\backslash\mathbb{A}^\times}_{=\mathbb{R}_{>0}GL_1(\mathbb{Q})\backslash GL_1(\mathbb{A}^\times)}\to\mathbb{C}^\times$

§ 1.2 Hecke L-functions

* F, $\mathcal{O}_F$, $\mathbb{A}=\mathbb{A}_F$, $\psi=\otimes_v\psi_v : F\backslash\mathbb{A}\to\mathbb{C}^1$

$\rightsquigarrow$ Fourier transform on $S(\mathbb{A})=\otimes_v' S(F_v)$ (restricted tensor product w.r.t. $1_{\mathcal{O}_\mathfrak{p}}$) & $S(F_v)$. $\hat{\Phi}(x)=\int_{\mathbb{A}}\Phi(y)\psi(-xy)\,dy$

with: $\begin{cases} dx_v \text{ on } F_v : \text{ self-dual Haar measure } \hat{\hat{\Phi}}_v(x_v)=\Phi(-x_v) \\ dx=\prod_v dx_v \text{ on } \mathbb{A} : \quad \hat{\hat{\Phi}}(x)=\Phi(-x) \end{cases}$

$\rightsquigarrow d^\times x_v = |x_v|_v^{-1}dx_v$ on $F_v^\times$ & $d^\times x = \prod_v \zeta_v(1)\dfrac{dx_v}{|x_v|_v}$ for $\zeta_v(s) := \begin{cases} \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) & F_v=\mathbb{R} \\ (2\pi)^{1-s}\Gamma(s) & F_v=\mathbb{C} \\ (1-N(\mathfrak{p})^{-s})^{-1} & v=\mathfrak{p}<\infty \end{cases}$

Tamagawa measure for GL_1 with convergence factors $\zeta_v(1)$

* $\chi=\otimes_v\chi_v : F^\times\mathbb{R}_{>0}\backslash\mathbb{A}^\times\to\mathbb{C}^1$ (or $F^\times\backslash\mathbb{A}^\times\to\mathbb{C}^1$) with $\chi_v : F_v^\times\to\mathbb{C}^1$ (or $\mathbb{C}^\times$)

For $v=\mathfrak{p}<\infty$, define $L_\mathfrak{p}(s,\chi_\mathfrak{p}) = \begin{cases} 1 & \text{if } \chi_\mathfrak{p}|_{\mathcal{O}_\mathfrak{p}^\times}\neq 1 \\ (1-\chi_\mathfrak{p}(\varpi_\mathfrak{p})\cdot N(\mathfrak{p})^{-s})^{-1} & \text{if } \chi_\mathfrak{p}|_{\mathcal{O}_\mathfrak{p}^\times}=1 \end{cases}$

$\varpi_\mathfrak{p}$: uniformizer s.t. $\mathfrak{p}\mathcal{O}_\mathfrak{p}=\varpi_\mathfrak{p}\mathcal{O}_\mathfrak{p}$

Then $L(s,\chi)=\prod_{\mathfrak{p}<\infty}L_\mathfrak{p}(s,\chi_\mathfrak{p})$ abs. conv. for $\mathrm{Re}\, s>1$

Goal: Meromorphic continuation of $L(s,\chi)$ & F.E.

* Global Theory: $\Phi\in S(\mathbb{A})$ & $\mathrm{Re}\, s>1$, $\chi : F^\times\mathbb{R}_{>0}\backslash\mathbb{A}^\times\to\mathbb{C}^1$

$$Z(s,\Phi,\chi) := \int_{\mathbb{A}^\times}\Phi(t)\chi(t)|t|_{\mathbb{A}}^s\,d^\times t = \int_{F^\times\backslash\mathbb{A}^\times}\Big(\sum_{\alpha\in F^\times}\Phi(\alpha t)\Big)\chi(t)|t|_{\mathbb{A}}^s\,d^\times t$$

Lemma 3: (Adelic Poisson) For any $\Phi\in S(\mathbb{A})$ we have

$$\sum_{\alpha\in F}\Phi(\alpha) = \sum_{\alpha\in F}\hat{\Phi}(\alpha)$$

$$Z(s,\Phi,\chi) = \int_{F^\times\backslash\mathbb{A}^\times}\Big(\sum_{\alpha}\Phi(\alpha t)\Big)\chi(t)|t|_{\mathbb{A}}^s\,d^\times t + \int_{F^\times\backslash\mathbb{A}^\times}\Big(\sum_{\alpha}\hat{\Phi}(\alpha t)\Big)\chi^{-1}(t)|t|_{\mathbb{A}}^{1-s}\,d^\times t$$

$$Z(s,\Phi,\chi) = \int_{|t|_{\mathbb{A}}\geq 1} \cdots + \int_{|t|_{\mathbb{A}}\geq 1} \cdots$$

$$+\ \delta_{\chi=\mathbb{1}} \operatorname{Vol}(F^\times \mathbb{R}_{>0}\backslash \mathbb{A}^\times)\left(\frac{\hat{\Phi}(0)}{s-1} - \frac{\Phi(0)}{s}\right)$$

The RHS is abs. conv. for all $s\in\mathbb{C}$ & meromorphic with symmetry

$$Z(1-s,\hat{\Phi},\chi^{-1}) = Z(s,\Phi,\chi)$$

* Local Theory[1/2]: Assume $\Phi = \otimes_v' \Phi_v$ factorizable. Then

$$Z(s,\Phi,\chi) = \prod_v Z_v(s,\Phi_v,\chi_v) \quad \text{factorizable, too.}$$

$$Z_v(s,\Phi_v,\chi_v) = \int_{F_v^\times} \Phi_v(t)\chi_v(t)|t|_v^s \, d^\times t$$

Unramified computation: At $v=\mathfrak{p}<\infty$ & $\Phi_\mathfrak{p} = \mathbb{1}_{\mathcal{O}_\mathfrak{p}}$ & $\chi_\mathfrak{p}|_{\mathcal{O}_\mathfrak{p}^\times} = \mathbb{1}$ & $\operatorname{Vol}(\mathcal{O}_\mathfrak{p}, dx_\mathfrak{p})=1$

$$Z_\mathfrak{p}(s,\Phi_\mathfrak{p},\chi_\mathfrak{p}) = \sum_{n=0}^{\infty} \chi_\mathfrak{p}(\varpi_\mathfrak{p})^n \cdot N(\mathfrak{p})^{-ns} = L_\mathfrak{p}(s,\chi_\mathfrak{p})$$

Rmk: (1) If $\operatorname{Vol}(\mathcal{O}_\mathfrak{p}, dx_\mathfrak{p})\neq 1$, then $\operatorname{Vol}(\mathcal{O}_\mathfrak{p}, dx_\mathfrak{p}) = D_\mathfrak{p}^{-\frac{1}{2}}$ with local abs. discriminant $D_\mathfrak{p}$

Then $\quad Z_\mathfrak{p}(s,\Phi_\mathfrak{p},\chi_\mathfrak{p}) = D_\mathfrak{p}^{-\frac{1}{2}} L_\mathfrak{p}(s,\chi_\mathfrak{p})$

(2) $$Z(s,\Phi,\chi) = L(s,\chi)\cdot \prod_{v|\infty} Z_v(s,\Phi_v,\chi_v)\cdot \prod_{\mathfrak{p}<\infty} \underbrace{\frac{Z_\mathfrak{p}(s,\Phi_\mathfrak{p},\chi_\mathfrak{p})}{L_\mathfrak{p}(s,\chi_\mathfrak{p})}}_{=1 \text{ a.e. } \mathfrak{p}} = L(s,\chi)\cdot \mathrm{Wt}(s,\Phi,\chi)$$

justifies "integral representation".

Local F.E.: $Z_v(s,\Phi_v,\chi_v) = \int_{F_v^\times} \Phi_v(t)\chi_v(t)|t|_v^s\, d^\times t$ abs. conv. for $\operatorname{Re} s>0$

Prop 1. For $0<\operatorname{Re} s<1$ & $\Phi_v, \Psi_v \in \mathcal{S}(F_v)$ we have

$$Z_v(s,\Phi_v,\chi_v)\, Z_v(1-s,\hat{\Psi}_v,\chi_v^{-1}) = Z_v(s,\Psi_v,\chi_v)\, Z_v(1-s,\hat{\Phi}_v,\chi_v^{-1})$$

Proof: $$\text{LHS} = \int_{F_v^\times}\int_{F_v^\times} \Phi_v(t_1)\hat{\Psi}_v(t_2)\chi_v(t_1t_2^{-1})|t_1t_2^{-1}|_v^s \cdot |t_2|_v \, d^\times t_1 \, d^\times t_2$$

$$= \zeta_v(1)^2 \int_{F_v}\left(\int_{F_v} \Phi_v(t_1)\hat{\Psi}_v(t_1t_2)\,dt_1\right)\chi_v(t_2)^{-1}|t_2|_v^{-s}\, dt_2 \qquad (*)$$

Plancherel for Fourier $\Rightarrow$ $$\int_{F_v}\Phi_v(t_1)\hat{\Psi}_v(t_1t_2)\,dt_1 = \int_{F_v}\hat{\Phi}_v(t_1)|t_2|^{-1}\Psi_v(t_1t_2^{-1})\,dt_1$$

$$= \int_{F_v}\Psi_v(t_1)\hat{\Phi}_v(t_1t_2)\,dt_1$$

Hence $(*)$ is symmetric for $\Phi_v \leftrightarrow \Psi_v$ $\Rightarrow$ LHS = RHS $\square$

Cor 2. There is $\gamma_v(s,\chi_v,\psi_v)\in\mathbb{C}$ independent of $\Phi_v\in\mathcal{S}(F_v)$ s.t.

$$Z_v(1-s,\hat{\Phi}_v,\chi_v^{-1}) = \gamma_v(s,\chi_v,\psi_v)\, Z_v(s,\Phi_v,\chi_v) \qquad 0<\operatorname{Re} s<1$$

* Local computation: Drop "v" for simplicity [RV99, §2.1]

Goal: Determine $\gamma(s,\chi,\psi)$

Reductions: $\left\{\begin{array}{l} F_1 := \{x\in F^\times \mid |x|_F = 1\}. \text{ Only need to consider } \chi: F_1\to\mathbb{C}^1 \text{ \& act. trivially to } F^\wedge \\ \gamma(s,\chi,a\psi) = \chi(a)|a|_F^{s-\frac{1}{2}}\gamma(s,\chi,\psi) \text{ for } a\in F^\times \text{ \& } a\psi(x) := \psi(ax)\end{array}\right.$

Add. char. à la Tate: $\mathrm{Tr} = \mathrm{Tr}_{F/\mathbb{Q}} : \mathbb{A}_F \to \mathbb{A}_{\mathbb{Q}}$, $\psi = \psi_0 \circ \mathrm{Tr}$ & $\psi_0 : \mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}} \to \mathbb{C}^1$

$$\psi_{0,v}(x) = \begin{cases} e^{2\pi i x}, & \text{if } \mathbb{Q}_v = \mathbb{R} \\ e^{-2\pi i \{x\}_p}, & \text{if } v = p \ \& \ \{x\}_p \in \mathbb{Z}[\tfrac{1}{p}] \text{ s.t. } x - \{x\}_p \in \mathbb{Z}_p \end{cases}$$

Exercise: Verify ψ_0 is trivial on $\mathbb{Q}$.

Real Case: ($F_v = \mathbb{R}$) $\psi(x) = e^{2\pi i x}$ $\chi \in \{\mathbb{1}, \mathrm{sgn}\}$ $\Phi(x) = \begin{cases} e^{-\pi x^2} & \text{if } \chi = \mathbb{1} \\ x e^{-\pi x^2} & \text{if } \chi = \mathrm{sgn} \end{cases}$

$$\Rightarrow \gamma(s,\chi,\psi) = \begin{cases} \Gamma_{\mathbb{R}}(1-s)/\Gamma_{\mathbb{R}}(s) & \text{if } \chi = \mathbb{1} \\ i \cdot \Gamma_{\mathbb{R}}(1-s+1)/\Gamma_{\mathbb{R}}(s+1) & \text{if } \chi = \mathrm{sgn} \end{cases} \quad \text{where } \Gamma_{\mathbb{R}}(s) := \pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})$$

Complex Case: ($F_v = \mathbb{C}$) $\psi(x) = e^{2\pi i (x+\bar{x})}$ $\chi \in \{\chi_n(x) = (\frac{x}{|x|})^n \mid n \in \mathbb{Z}\}$

$$\Phi(x) = \begin{cases} \bar{x}^n e^{-2\pi x \bar{x}} & \text{if } \chi = \chi_n \ \& \ n \geq 0 \\ x^{-n} e^{-2\pi x \cdot \bar{x}} & \text{if } \chi = \chi_n \ \& \ n < 0 \end{cases}$$

$$\Rightarrow \gamma(s,\chi_n,\psi) = i^{|n|}\, \Gamma_{\mathbb{C}}(1-s+\tfrac{|n|}{2}) / \Gamma_{\mathbb{C}}(s+\tfrac{|n|}{2}) \quad \text{where } \Gamma_{\mathbb{C}}(s) := (2\pi)^{1-s}\Gamma(s)$$

Rmk: $\Gamma_{\mathbb{R}}(s)$ resp. $\Gamma_{\mathbb{C}}(s)$ coincides with $\zeta_v(s)$ defined before

Non-archimedean Case: ($v = \mathfrak{p} < \infty$) By **Reductions**, can assume $\psi|_{\mathcal{O}} = \mathbb{1}$ & $\psi|_{\mathfrak{p}^{-1}} \neq \mathbb{1}$ $a(\psi) = 0$ Generally $\psi|_{\mathfrak{p}^{-n}} = \mathbb{1}$, $\psi|_{\mathfrak{p}^{-n-1}} \neq \mathbb{1}$ $\Leftrightarrow a(\psi) = n$ & $c(\psi$

Write $U_n = (1+\varpi^n\mathcal{O}) \cap \mathcal{O}^\times$ & $\chi|_{U_n} = \mathbb{1}$ but $\chi|_{U_{n-1}} \neq \mathbb{1}$. $a(\chi) = n$ & $c(\chi) = q^n$

Take $\Phi(x) = \begin{cases} \mathbb{1}_{\mathcal{O}}(x) & \text{if } n = 0 \\ \mathbb{1}_{U_n}(x) & \text{if } n > 0 \end{cases} \Rightarrow \hat{\Phi}(x) = \begin{cases} \mathbb{1}_{\mathcal{O}}(x) & \text{if } n = 0 \\ q^{-n}\psi(-x)\,\mathbb{1}_{\varpi^{-n}\mathcal{O}}(x) & \text{if } n > 0 \end{cases}$ $q := N_v(\mathfrak{p})$

$$\Rightarrow \gamma(s,\chi,\psi) = \begin{cases} (1-\chi(\varpi)q^{-s})/(1-\chi^{-1}(\varpi)q^{-(1-s)}) = L(1-s,\chi^{-1})/L(s,\chi) & \text{if } n = 0 \\ q^{n(\frac{1}{2}-s)} \cdot g(\chi^{-1},\psi^{-1})\, q^{\frac{n}{2}} & \text{if } n > 0 \end{cases}$$

Def 1. For χ with $n = c(\chi) > 0$ & ψ with $c(\psi) = 0$, the Gauss sum

$$g(\chi,\psi) := \int_{\mathcal{O}-\mathfrak{p}} \psi(\varpi^{-n}t)\,\chi(\varpi^{-n}t)\,dt$$

Prop 2. (1) $g(\chi,\psi)$ is well-defined, i.e., independent of choice of ϖ

(2) $|g(\chi,\psi)| = q^{-\frac{n}{2}}$ & $g(\chi,\psi)\,g(\chi^{-1},\psi) = \chi(-1)\,q^{-n}$

Proof: (1) obvious. (2) Easy to see $\overline{g(\chi,\psi)} = \chi(-1)\,g(\chi^{-1},\psi)$. Only need to prove the first equation.

$$|g(\chi,\psi)|^2 = \iint_{(\mathcal{O}^\times)^2} \psi(\varpi^{-n}(t_1-t_2))\,\chi(t_1t_2^{-1})\,dt_1\,dt_2$$

$$= \iint_{(\mathcal{O}^\times)^2} \psi(\varpi^{-n}t_2(t_1-1))\,\chi(t_1)\,dt_1\,dt_2$$

Note $\int_{\mathcal{O}^\times} \psi(\varpi^{-n}(t_1-1)t_2)\,dt_2 = \int_{\mathcal{O}} \psi(\varpi^{-n}(t_1-1)t_2)\,dt_2 - q^{-1}\int_{\mathcal{O}} \psi(\varpi^{1-n}(t_1-1)t_2)\,dt_2$

$$= \mathbb{1}_{1+\mathfrak{p}^n}(t_1) - q^{-1}\,\mathbb{1}_{1+\mathfrak{p}^{n-1}}(t_1) \quad \Rightarrow$$

$$|g(\chi,\psi)|^2 = \int_{1+\mathfrak{P}^n} \chi(t_1)\,dt_1 - q^{-1}\int_{1+\mathfrak{P}^{n-1}} \chi(t_1)\,dt_1 = q^{-n} \qquad \square$$

Thm 2. (1) The local γ-factors have the form

$$\gamma_v(s,\chi_v,\psi_v) = \varepsilon_v(s,\chi_v,\psi_v)\cdot\frac{L_v(1-s,\chi_v^{-1})}{L_v(s,\chi_v)}$$

where the local ε-factors have the form

$$\varepsilon_v(s,\chi_v,\psi_v) = [C(\chi_v)\,C(\psi_v)]^{\frac{1}{2}-s}\,\varepsilon_v(\chi_v,\psi_v)$$

for some local root number $\varepsilon_v(\chi_v,\psi_v)\in\mathbb{C}^1$.

(2) The local zeta integrals have meromorphic continuation & F.E.

$$Z_v(1-s,\widehat{\Phi}_v,\chi_v^{-1}) = \gamma_v(s,\chi_v,\psi_v)\,Z_v(s,\Phi_v,\chi_v) \quad \text{for } s\in\mathbb{C}$$

Exercise: Prove meromorphic continuation of $Z_v(s,\Phi_v,\chi_v)$ by truncation ($|t|_v\geq 1$ vs. $|t|_v<1$) as in the global case & asymptotic behavior at $|t|_v\to 0$.