Rankin–Selberg GL(2) × GL(1)

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§4.1 Tate's Thesis revisited

Recall the Euler decomposition of Tate's zeta integral

$$\int_{F^\times \backslash \mathbb{A}^\times} \Big(\sum_{\alpha \in F^\times} \Phi(\alpha t) \Big) \chi(t) |t|_{\mathbb{A}}^{s} d^\times t$$
$$\Vert$$
$$Z(s, \Phi, \chi) = L(s, \chi) \cdot \prod_{v | \infty} Z_v(s, \Phi_v, \chi_v) \cdot \prod_{p < \infty} \frac{Z_p(s, \Phi_p, \chi_p)}{L_p(s, \chi_p)}$$

Weil's re-interpretation [We 65] of local F.E.s in terms of uniqueness of (homogeneous) tempered distributions:

$\Phi_v \mapsto Z_v(s, \Phi_v, \chi_v)$ belongs to $\mathrm{Hom}_{F_v^\times}(\mathcal{S}(F_v), \chi_v^{-1} |\cdot|_v^{-s})$

- action of $F_v^\times$ on $S(F_v)$: $t.\Phi(x) := \Phi(xt)$.
- a character χ_v of $F_v^\times$ is considered as one dim'l rep. of $F_v^\times$.

In fact $Z_v(s, t.\Phi_v, \chi_v) = \int_{F_v^\times} \Phi_v(xt)\chi_v(x)|x|_v^s d^\times x = \chi_v^{-1}(t)|t|_v^{-s} \cdot Z_v(s, \Phi_v, \chi_v)$

Prop. 1. For χ_v unitary & $\mathrm{Re}\, s > 0$ we have $\dim_{\mathbb{C}} \mathrm{Hom}_{F_v^\times}(S(F_v), \chi_v^{-1}|\cdot|_v^{-s}) = 1$.

Rmk: For $F_v = \mathbb{R}$ this is uniqueness of homogeneous distributions.

Sketch of Proof: In the archimedean case try to deduce the differential equation satisfied by a $D \in \mathrm{Hom}_{F_v^\times}(S(F_v), \chi_v^{-1}|\cdot|_v^{-s})$ & solve it on each connected component of $F_v^\times$. Then show the unique way of gluing the solutions by taking the action of $F_v^1 := \{t \in F_v^\times \mid |t|_v = 1\}$ into account.

Lemma 1. $\Phi_v \mapsto Z_v(1-s, \hat{\Phi}_v, \chi_v^{-1})$ also belongs to $\mathrm{Hom}_{F_v^\times}(S(F_v), \chi_v^{-1}|\cdot|_v^{-s})$ for $0 < \mathrm{Re}\, s < 1$.

Proof: $\widehat{t.\Phi_v} = t^{-1}.\hat{\Phi}_v \cdot |t|^{-1} \Rightarrow$

$$Z_v(1-s, \widehat{t.\Phi_v}, \chi_v^{-1}) = |t|^{-1} Z_v(1-s, t^{-1}.\hat{\Phi}_v, \chi_v^{-1}) = |t|^{-1} \cdot \chi_v(t^{-1})|t^{-1}|_v^{s-1} Z_v(1-s, \hat{\Phi}_v, \chi_v^{-1})$$
$$= \chi_v^{-1}(t)|t|_v^{-s} \cdot Z_v(1-s, \hat{\Phi}_v, \chi_v^{-1}) \qquad \square$$

Rmk: Godement-Jacquet's theory generalizes Tate's thesis beautifully in terms of applications of Fourier analysis, but does not capture the uniqueness property of distributions in Weil's reinterpretation. To find a good analogue from this point of view, the uniqueness of Whittaker functionals is the key.

§4.2 Whittaker Functionals & Functions

* Archimedean Case

(π, V) unitary irred. of $GL_2(\mathbb{R})$ or $GL_2(\mathbb{C})$. Let $V^\infty \subset V$ be the subspace of smooth vectors, topologized by seminorms $v \mapsto \|\pi(D).v\|$, $D \in U(\mathfrak{gl}_2(\mathbb{R}))$ or $U(\mathfrak{gl}_2(\mathbb{C}))$. This make V^∞ a Fréchet space.

Def. 1. A Whittaker functional on (π^∞, V^∞) is a smooth linear functional

$\lambda: V^\infty \to \mathbb{C}$ s.t. $\lambda(\pi(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}).v) = \psi(x)\lambda(v)$ for all $x \in F_v$.

Thm 1. (Shalika) The dimension of Whittaker functionals is at most 1.

Shalika [Sh 74] proved various multiplicity one results based on some multiplicity one result of distributions on the groups $GL_n(F)$.

Bump [Bu 98] illustrated the case for GL_2 for the $(\mathfrak{g}, K)$-modules instead of V^∞ in §2.8.

* Non-archimedean Case

(π, V) unitary irreducible of $GL_2(F)$. Let $V^\infty \subset V$ be the subspace of K-finite vectors (= smooth vectors).

Def. 2. A Whittaker functional on (π^∞, V^∞) is a linear functional $\lambda: V^\infty \to \mathbb{C}$ s.t.

$$\lambda(\pi(\begin{smallmatrix}1 & x\\ & 1\end{smallmatrix}).v) = \psi(x)\lambda(v) \quad \text{for all } v \in V^\infty \text{ \& } x \in F.$$

Thm 2. The space of Whittaker functionals has dimension at most one.

Bump [Bu 98] gives a complete proof for GL_2 in §4.4.

* Global Case

Let (π, V) be unitary irreducible automorphic representation appearing in the spectral decomposition of $L^2(GL_2, \omega)$. Note that in the case of continuous spectrum we understand (π, V) in the induced model of global principal series, and only the smooth vectors $f \in V^\infty$ has realization as smooth functions on $GL_2(F)\backslash GL_2(\mathbb{A})$ via the Eisenstein intertwiner $E: V^\infty \to C^\infty(GL_2, \omega)$.

Note that in the case of cuspidal representations $(\pi, V) \subset L_0^2(GL_2, \omega)$ the smooth vectors in $V^\infty \subset C^\infty(GL_2, \omega)$ are automatically smooth functions by the Sobolev embedding theorem since $GL_2(F)\backslash GL_2(\mathbb{A})$ is locally Euclidean as a smooth manifold.

Def. 3. In the cuspidal case, a Whittaker functional is given by

$$\lambda: V^\infty \to \mathbb{C}, \quad \lambda(\varphi) := \int_{F\backslash\mathbb{A}} \varphi\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right) \psi(-x)\, dx$$

Def. 3. (G's) In the cuspidal case, a Whittaker functional is given by

$$\lambda: V^\infty \to \mathbb{C}, \quad \lambda(f) := \int_{F\backslash\mathbb{A}} E(f)\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right) \psi(-x)\, dx .$$

* Fourier-Whittaker Expansion

Def. 4. In both local & global cases, given a Whittaker functional λ we call the function $W_v(g) := \lambda(\pi(g).v)$ the Whittaker function of the smooth vector $v \in V^\infty$. All such functions form the Whittaker model $W(\pi^\infty, \psi)$ of (π^∞, V^∞).

In the case of cuspidal (π, V), we have for $\varphi \in V^\infty$ & $\alpha \in F^\times$

$$\begin{aligned} W_\varphi\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix} g\right) &= \int_{F\backslash\mathbb{A}} \varphi\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\alpha & \\ & 1\end{pmatrix} g\right) \psi(-x)\, dx \\ &= \int_{F\backslash\mathbb{A}} \varphi\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix}\begin{pmatrix}1 & \alpha^{-1}x\\ & 1\end{pmatrix} g\right) \psi(-x)\, dx \\ &= \int_{F\backslash\mathbb{A}} \varphi\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix} g\right) \psi(-\alpha x)\, dx \end{aligned}$$

by the left invariance by $GL_2(F)$ of φ. Hence $W_\varphi\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix} g\right)$ is simply the Fourier coefficient of the (smooth) function

$$F\backslash\mathbb{A} \to \mathbb{C}, \quad u \mapsto \varphi\left(\begin{pmatrix}1 & u\\ & 1\end{pmatrix} g\right).$$

By the Fourier inversion formula on $F\backslash\mathbb{A}$, one gets

$$\varphi(g) = \sum_{\alpha \in F^\times} W_\varphi\left(\begin{pmatrix}\alpha & \\ & 1\end{pmatrix} g\right). \qquad (1)$$

Rmk: The Whittaker functions offer a concrete way to find decomposable vectors. Namely, φ is decomposable iff $W_\varphi(g) = \prod_v W_{\varphi_v}(g_v)$, $g = (g_v)$.

Rmk: The rapid decay of φ (in a Siegel domain) implies that of W_φ.

* Local Bounds

Thm. 3. (1) In the archimedean case we have for any $N \in \mathbb{Z}_{\geq 0}$

$$|W_v(g)| \leq \mathrm{Ht}_v(g)^{-N} \cdot S_N(v) \qquad \forall\, v \in V^\infty \ \&\ \text{as } \mathrm{Ht}_v(g) \to +\infty$$

where – $\mathrm{Ht}_v\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}t & \\ & 1\end{pmatrix}k\right) := |t|_v$ is the local height function

– $S_N(\cdot)$ is a seminorm which depends only on N.

(2) In the non-archimedean case the support of the function

$$t \mapsto W_v\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}yt & \\ & 1\end{pmatrix}k\right)$$ has bounded image under the norm

$|\cdot|_v$ map, which depends only on v, not on $x\in F$, $y\in F^\times$, $k\in GL_2(\mathcal{O})$.

Rmk: The non-archimedean case is an easy exercise while the archimedean case is a bit more difficult. A proof can be found in [CPS 90] Lemma I.1.2 & [Wu 14] §2.6.

Thm 4. There is a continuous semi-norm S and $d\geq 0$ s.t.

$$|W_v(g)| \leq \mathrm{Ht}_v(g)^{\frac{1}{2}-\theta}(1+|\log \mathrm{Ht}_v(g)|)^d\, S(v) \quad \text{for all } v\in V^\infty$$

if (π, V) is θ-tempered.

Rmk: (1) The non-archimedean case follows from the theory of Jacquet modules and is easy.

(2) The tempered case in the archimedean case is attributed to Wallach [Wa 88&92] by Jacquet [Ja 09] §3.4. The non-tempered case can be deduced from the integral representations of Whittaker functions.

(3) The point here is that these estimations are as fundamental as the decay of matrix coefficients, since the Whittaker functions

$$W_v(g) = \lambda(\pi(g).v) = \langle \pi(g).v, \lambda\rangle$$

are understood as generalized matrix coefficients.

§ 4.3. Rankin-Selberg Theory : Global Theory

Let $(\pi, V) \subset L^2_0(GL_2, \omega)$ be cuspidal, $\chi: F^\times\mathbb{R}_{>0}\backslash\mathbb{A}^\times \to \mathbb{C}^1$ & $\varphi\in V^\infty$. Recall that φ has rapid decay in any Siegel domain. The global zeta integral is defined to be

$$Z(s,\varphi,\chi) := \int_{F^\times\backslash\mathbb{A}^\times} \varphi(a(t))\,\chi(t)\,|t|_{\mathbb{A}}^{s-\frac{1}{2}}\,d^\times t$$

Prop. 1 The integral $Z(s,\varphi,\chi)$ is absolutely convergent for all $s\in\mathbb{C}$. It satisfies

$$Z(1-s,\pi(w).\varphi,\omega^{-1}\chi^{-1}) = Z(s,\varphi,\chi) \quad \text{where } w=\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \text{ or } \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

Proof: By the left invariance by $GL_2(F)$ (hence w) of φ we have

$$\varphi(a(t)) = \varphi(w a(t) w^{-1} w) = \omega(t)\,\varphi(a(t^{-1})w) \quad \Rightarrow$$

$$Z(s,\varphi,\chi) = \int_{|t|_{\mathbb{A}}\geq 1} \varphi(a(t))\,\chi(t)\,|t|_{\mathbb{A}}^{s-\frac{1}{2}}\,d^\times t + \int_{|t|_{\mathbb{A}}\leq 1} \varphi(a(t^{-1})w)\,\omega\chi(t)\,|t|_{\mathbb{A}}^{s-\frac{1}{2}}\,d^\times t$$

$$= \int_{|t|_{\mathbb{A}}\geq 1} \varphi(a(t))\,\chi(t)\,|t|_{\mathbb{A}}^{s-\frac{1}{2}}\,d^\times t + \int_{|t|_{\mathbb{A}}\geq 1} \pi(w).\varphi(a(t))\,(\omega\chi)^{-1}(t)\,|t|_{\mathbb{A}}^{\frac{1}{2}-s}\,d^\times t$$

Each integral is abs. conv. for all $s\in\mathbb{C}$ by rapid decay of φ & $\pi(w).\varphi$.

Moreover, the two integrals are flipped by the change $s\leftrightarrow 1-s$, $\varphi\leftrightarrow\pi(w).\varphi$ & $\chi\leftrightarrow\omega^{-1}\chi^{-1}$.

Hence we get the desired functional equation. $\square$

If we insert the Fourier-Whittaker expansion (1), we get for decomposable φ

$$Z(s,\varphi,\chi) = \int_{F^\times\backslash\mathbb{A}^\times}\Big(\sum_{\alpha\in F^\times} W_\varphi(a(\alpha t))\Big)\,\chi(t)\,|t|_{\mathbb{A}}^{s-\frac{1}{2}}\,d^\times t$$

$$= \int_{\mathbb{A}^\times} W_\varphi(a(t))\,\chi(t)\,|t|_{\mathbb{A}}^{s-\frac{1}{2}}\,d^\times t$$

$$= \prod_v \int_{F_v^\times} W_{\varphi,v}(a(t))\,\chi_v(t)\,|t|_v^{s-\frac{1}{2}}\,d^\times t =: \prod_v Z_v(s,W_{\varphi,v},\chi_v)$$

The abs. conv. of the last expression can be verified for $\Re s \gg 1$ by the local bounds of the Whittaker function Thm. 3. & Thm. 4., together with the computation at the unramified cases/places.

Rmk: The above construction of zeta-integral is analogue of Hecke's one in the classical setting $L(s,f) := \int_0^\infty f(iy)\,y^{s+\frac{k}{2}}\,\frac{dy}{y}$ for $f\in S_k^{new}(\Gamma_0(N))$.

For details of comparison, see [Ge75] §6.D.

Rmk: The generalization to Eisenstein series requires a modification of the definition $Z(s,\varphi,\chi) := \int_{F^\times\backslash\mathbb{A}^\times} (\varphi-\varphi_N)(a(t))\,\chi(t)\,|t|_{\mathbb{A}}^{s-\frac{1}{2}}\,d^\times t$

which is abs. conv. for $\Re s\gg 1$. We have the analogous expression

$$Z(s,\varphi,\chi) = \int_{|t|_{\mathbb{A}}\geq 1} (\varphi-\varphi_N)(a(t))\,\chi(t)\,|t|_{\mathbb{A}}^{s-\frac{1}{2}}\,d^\times t$$

$$+ \int_{|t|_{\mathbb{A}}\geq 1} (\pi(w)\varphi - \pi(w)\varphi_N)(a(t))\,\omega^{-1}\chi^{-1}(t)\,|t|_{\mathbb{A}}^{\frac{1}{2}-s}\,d^\times t$$

$$- \int_{|t|_{\mathbb{A}} \le 1} \mathcal{F}_N(a(t))\, \chi(t) |t|_{\mathbb{A}}^{s-\frac{1}{2}}\, d^\times t$$

$$+ \int_{|t|_{\mathbb{A}} \le 1} \pi(w).\mathcal{F}_N(a(t^{-1}))\, \omega\chi(t) |t|_{\mathbb{A}}^{s-\frac{1}{2}}\, d^\times t$$

- abs. conv. Bessel!
- explicitly computable & admit merom. cont.

§4.4 Rankin-Selberg Theory: Local Theory

* The local theory (for K-finite vectors) occupies the whole chapter I of [JL70]. See also [Bu98] §4.7 for the non-archimedean case. The viewpoint follows Weil's re-interpretation of Tate's thesis as follows. We omit subscript v for simplicity.

$\mathcal{W}(\pi^\infty, \psi) \to \mathbb{C}$, $W \mapsto Z(s, W, \chi)$ is a linear functional

$Z(s, \cdot, \chi) \in \mathrm{Hom}_{F^\times}(\pi^\infty, \chi^{-1}|\cdot|^{-(s-\frac{1}{2})})$, which has dimension ≤ 1 for all but at most 2 exceptional values of s. The key to see this multiplicity one is the Kirillov theory, which we summarize as the following theorem.

Thm. 5. Let $\mathcal{K}(\pi^\infty, \psi) := \{ K : F^\times \to \mathbb{C} \mid K(t) = W(a(t)) \text{ for some } W \in \mathcal{W}(\pi^\infty, \psi) \}$.

Then the restriction map $\mathcal{W}(\pi^\infty, \psi) \to \mathcal{K}(\pi^\infty, \psi)$, $W \mapsto (t \mapsto W(a(t)))$ is injective. ($\mathcal{K}(\pi^\infty, \psi)$ is called the Kirillov model of π^∞.)

It contains the unique irreducible representation of infinite dimension τ of the mirabolic group $P_1(F) := \{ \begin{pmatrix} t & x \\ 0 & 1 \end{pmatrix} \mid t \in F^\times, x \in F \}$ precisely once.

Moreover, $\mathcal{K}(\pi^\infty, \psi) / C_c^\infty(F^\times)$ has dimension ≤ 2 & it has dimension 0 (i.e. $\mathcal{K}(\pi^\infty, \psi) = C_c^\infty(F^\times)$) iff π is supercuspidal.

Rmk: The representation τ is realized in $C_c^\infty(F^\times)$ by

$$\left(\tau\begin{pmatrix} t & x \\ & 1 \end{pmatrix}.f\right)(y) := \psi(xy) f(yt).$$

Now the desired multiplicity one follows from (Compare Prop 1.)

$$\dim_{\mathbb{C}} \mathrm{Hom}_{F^\times}(C_c^\infty(F^\times), \chi^{-1}|\cdot|^{-s+\frac{1}{2}}) = 1 \quad \text{for all } s \in \mathbb{C}.$$

Finally we check easily that $W \mapsto Z(1-s, \pi(w).W, \omega^{-1}\chi^{-1})$ also belongs to $\mathrm{Hom}_{F^\times}(\pi^\infty, \chi^{-1}|\cdot|^{-s+\frac{1}{2}})$ & concludes the existence of local F.E..

Rmk: The representatives of $\mathcal{K}(\pi^\infty, \psi) / C_c^\infty(F^\times)$ can be explicitly determined as $\chi_1(t)\Phi_1(t)$ & $\chi_2(t)\Phi_2(t)$ for some characters χ_1, χ_2 of $F^\times$ determined by π & $\Phi_j \in \mathcal{S}(F)$. These are responsible for the existence of meromorphic continuation (say in the non-supercuspidal case).

* The equivalence between the Godement-Jacquet theory for $\pi \otimes \chi$ and the Rankin-Selberg theory for $\pi \times \chi$ is shown in [JPSS 79] §4 & 11. Note that we may easily reduce the equivalence to the case $\chi = \mathbb{1}$ is the trivial character. The following integrals played the pivot rôle:

$$Z(s, \Phi, W) := \int_{GL_2(F)} \Phi(g)\, W(g)\, |\det g|^{s+\frac{1}{2}} dg, \quad \Phi \in \mathcal{S}(M_2(F)),\ W \in \mathcal{W}(\pi^\infty, \psi).$$

- On the one hand we have $W(g) = \langle \pi(g).v_1, \lambda \rangle$ for a Whittaker functional $\lambda \in \widehat{V^\infty}$. Now that $\mathcal{S}(M_2(F))$ is a smooth Fréchet representation of $GL_2(F)$, the Dixmier-Malliavin's theorem shows that $\exists\ \Phi_i \in \mathcal{S}(M_2(F))$, $f_i \in C_c^\infty(G^1)$

$\{g \in GL_2(F) \mid |\det g| = 1\}$

s.t. $\Phi = \sum_{i=1}^{d} \int_{GL_2(F)} f_i(g)\, \Phi_i . g$. Hence

$$\begin{aligned} Z(s, \Phi, W) &= \sum_{i=1}^{d} \int_{GL_2(F)} \left(\int_{G^1} f_i(x)\, \Phi_i(xg)\, dx \right) \langle \pi(g).v_1, \lambda \rangle\, |\det g|^{s+\frac{1}{2}} dg \\ &= \sum_{i=1}^{d} \int_{GL_2(F)} \Phi_i(g) \left(\int_{G^1} f_i(x) \langle \pi(x^{-1})\pi(g).v_1, \lambda \rangle dx \right) |\det g|^{s+\frac{1}{2}} dg \\ &= \sum_{i=1}^{d} \int_{GL_2(F)} \Phi_i(g) \langle \pi(g).v_1, \widehat{\pi}(f_i)\lambda \rangle\, |\det g|^{s+\frac{1}{2}} dg \qquad (2) \end{aligned}$$

By continuity we have $\widehat{\pi}(f_i)\lambda \in (\widehat{V})^\infty$, thus each function

$$g \mapsto \langle \pi(g) v_1, \widehat{\pi}(f_i)\lambda \rangle$$

is a matrix coefficient in the usual sense. Thus each summand in (2) is a Godement-Jacquet zeta integral in the usual sense.

- On the other hand, we may relate $Z(s, \Phi, W)$ with $Z(s, W, \mathbb{1})$ as follows. For $\Phi \in \mathcal{S}(M_2(F))$ define functions (on $F^3 \times K^1$, $K^1 := K \cap SL_2(F)$)

$$\Theta_\Phi(u, u_2, v_2; k) := \int_F \Phi\left(\begin{pmatrix} u & x \\ & u_2 \end{pmatrix} k \right) \psi(u_2 x)\, dx$$

$$K_\Phi(v, u_2, v_2; k) := \int_F \Theta_\Phi(u, u_2, v_2; k)\, \psi(-vu)\, du$$

Schwartz-Bruhat in the first 3 variables!

Def. 5. We define a complex measure ℓ_Φ on $SL_2(F)$ by

$$\int_{SL_2(F)} f(h)\, d\ell_\Phi(h) = \int_{F^\times}\int_F\int_{K'} f\left(\begin{pmatrix} a_2^{-1} & v \\ & a_2 \end{pmatrix} x\right) K_\Phi(v, a_2, a_2^{-1} : x) \cdot |a_2|\, d^\times a_2\, dv\, dx.$$

Lemma 2. If H is a smooth function on $GL_2(F)$ s.t. $H\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = \psi(x) H(g)$

& $\int_{GL_2(F)} H(g)\Phi(g)\, dg$ is abs. convergent. Then

$$\int_{GL_2(F)} H(g)\Phi(g)\, dg = \int_{F^\times}\int_{SL_2(F)} H\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} h\right) |a|^{-1} d^\times a\, d\ell_\Phi(h).$$

Proof. We take the Iwasawa decomposition $g = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\begin{pmatrix} a_2^{-1} & v \\ & a_2 \end{pmatrix} x$, $x \in K'$

$$\Rightarrow \int_{GL_2(F)} H(g)\Phi(g)\, dg = \int_{K'}\int (H\cdot\Phi)\left(\begin{pmatrix} a_1 & x \\ & a_2 \end{pmatrix} x\right) dx\, |a_1|^{-1} d^\times a_1\, d^\times a_2\, dx$$

$$= \int_{K'}\int H\left(\begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} x\right)\left(\int_F \Phi\left(\begin{pmatrix} a_1 & x \\ & a_2 \end{pmatrix} x\right)\psi(x a_2^{-1})\, dx\right) |a_1|^{-1} d^\times a_1\, d^\times a_2\, dx$$

$$= \int_{K'}\int H\left(\begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} x\right) \Theta_\Phi(a_1, a_2, a_2^{-1} : x)\, |a_1|^{-1} d^\times a_1\, d^\times a_2\, dx$$

$$= \int_{K'}\int\left(\int_F H\left(\begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} x\right)\psi(v a_2) K_\Phi(v, a_2, a_2^{-1} : x)\, dv\right) |a_1|^{-1} d^\times a_1\, d^\times a_2\, dx$$

$$= \int_{K'}\int\left(\int_F H\left(\begin{pmatrix} a_1 & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & v \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & a_2 \end{pmatrix} x\right) K_\Phi(v, a_2, a_2^{-1} : x)\, dv\right) |a_1|^{-1} d^\times a_1\, d^\times a_2\, dx$$

Making the change of variable $a_1 \mapsto a_1 a_2^{-1}$ we get the desired equality! $\square$

Apply Lemma 2. with $H(g) = W(g)\, |\det g|^{s+\frac{1}{2}}$ we get

$$Z(s, \Phi, W) = \int_{F^\times}\left(\int_{SL_2(F)} W\left(\begin{pmatrix} t & \\ & 1 \end{pmatrix} h\right) d\ell_\Phi(h)\right) |t|^{s-\frac{1}{2}}\, d^\times t$$

$$= Z(s, W * \ell_\Phi, \mathbb{1})$$

which is a Rankin-Selberg integral for the smooth $W * \ell_\Phi \in \mathcal{W}(\pi^\infty, \psi)$.