§4.6 Tychonoff's Theorem

In this section, we give the proof of Tychonoff's theorem.

Theorem 4.6.1 (Tychonoff) The product of arbitrarily many compact spaces is compact.

Recall

1. A collection $\mathscr{A}$ of subsets of $\mathbb{X}$ satisfies the finite intersection property (f.i.p.) if the intersection of finitely many members of $\mathscr{A}$ is non-empty.

2. $\mathbb{X}$ is compact $\Leftrightarrow$ $\forall \mathscr{A} \subseteq \{\text{closed sets in } \mathbb{X}\}$ satisfying f.i.p., $\bigcap_{A \in \mathscr{A}} A \neq \phi$ $\Leftrightarrow$ $\forall \mathscr{A} \subseteq \mathcal{P}(\mathbb{X})$ satisfying f.i.p., $\bigcap_{A \in \mathscr{A}} C(A) \neq \phi$.

3. Zorn's lemma: Let $(\mathbb{X}, \leq)$ be a partial ordered set. If every linearly ordered subset of $\mathbb{X}$ has an upper bound in $\mathbb{X}$, then $\mathbb{X}$ has a maximal element.

- Linearly orderd subset $Y \subseteq \mathbb{X}$: $\forall y_1, y_2 \in Y$, we have $y_1 \leq y_2$ or $y_2 \leq y_1$.
- Upper bound of $Y \subseteq \mathbb{X}$: An element $u \in \mathbb{X}$ s.t. $y \leq u$ for $\forall y \in Y$.
- Maximal element: An element $m \in \mathbb{X}$ s.t. $x \geq m \Rightarrow x = m$.

Notation. Since we will next deal with sets involving multiple levels, we adopt the following conventions of notations:

1) a: elements of $\mathbb{X}$; 2) A: $A \subseteq \mathbb{X}$; 3) $\mathscr{A}$: $\mathscr{A} \subseteq \mathcal{P}(\mathbb{X})$;

4) $\mathbb{A}$: a collection of collections of subsets of A, i.e. $\mathbb{A} = \{\mathscr{A} \subseteq \mathcal{P}(\mathbb{X}) \mid \cdots\}$.

Lemma 4.6.2 Let $\mathbb{X}$ be a set. Let $\mathbb{F} = \{\mathscr{A} \subseteq \mathcal{P}(\mathbb{X}) \mid \mathscr{A} \text{ satisfies f.i.p.}\}$. Then $\forall \mathscr{A} \in \mathbb{F}$, $\exists \mathcal{U} \in \mathbb{F}$ s.t. $\mathscr{A} \subseteq \mathcal{U}$ and $\mathcal{U}$ is maximal in $\mathbb{F}$. (w.r.t. the inclusion order).

Proof. Given $\mathscr{A} \in \mathbb{F}$. Let $\mathbb{B} = \{\mathscr{B} \in \mathbb{F} \mid \mathscr{A} \subseteq \mathscr{B}\}$. We shall apply Zorn's lemma to find a maximal element of $\mathbb{B}$. Let $\mathbb{D} \subseteq \mathbb{B}$ be a linearly ordered subset of $\mathbb{A}$. We claim that $\mathcal{U} \triangleq \bigcup_{\mathscr{D} \in \mathbb{D}} \mathscr{D}$ is an upper bound of $\mathbb{D}$ in $\mathbb{B}$

It suffices to verify that $\mathcal{U}$ satisfies f.i.p.. Let $D_1, \cdots, D_n \in \mathcal{U}$. Then $1 \leq i \leq n$, $\exists \mathcal{D}_i \in \mathbb{D}$ s.t. $D_i \in \mathcal{D}_i$. But $\mathbb{D}$ is linearly ordered. So $\exists\, 1 \leq i_0 \leq n$ s.t. $\mathcal{D}_i \subseteq \mathcal{D}_{i_0}$ for all $1 \leq i \leq n$. Thus all of $D_1, \cdots, D_n$ are members of $\mathcal{D}_{i_0}$. Since $\mathcal{D}_{i_0}$ itself satisfies f.i.p.,

$$D_1 \cap \cdots \cap D_n \neq \emptyset.$$

hence $\mathcal{U}$ satisfies f.i.p..

By Zorn's lemma, $\mathbb{B}$ has a maximal element, which is what we want. □

Lemma 4.6.3. Let $\mathbb{X}$ be a set. Let $\mathbb{F} = \{\mathcal{A} \subseteq \mathcal{P}(\mathbb{X}) \mid \mathcal{A}$ satisfies f.i.p.$\}$. Let $\mathcal{M}$ be a maximal element of $\mathbb{F}$. Then

i) $\mathcal{M}$ is closed under finite intersections ($A_1, \cdots, A_n \in \mathcal{M} \Rightarrow A_1 \cap \cdots \cap A_n \in \mathcal{M}$).

ii) If $A \subseteq \mathbb{X}$ intersects every element of $\mathcal{M}$, then $A \in \mathcal{M}$.

Proof. i) Let $A_1, \cdots, A_n \in \mathcal{M}$ and let $B = A_1 \cap \cdots \cap A_n$. We show that $\mathcal{M} \cup \{B\}$ satisfies f.i.p. hence $B \in \mathcal{M}$ since $\mathcal{M}$ is maximal. Take finitely many members of $\mathcal{M}$. If none of them is B, there is nothing to prove. Otherwise, their intersection is of the form

$$D_1 \cap D_2 \cap \cdots \cap D_m \cap B = D_1 \cap D_2 \cap \cdots \cap D_m \cap A_1 \cap \cdots \cap A_n,$$

which is an intersection of elements of $\mathcal{M}$ hence is not empty.

ii) is similar. We show that $\mathcal{M} \cup \{A\}$ satisfies f.i.p.. Let $D_1, \cdots, D_n \in \mathcal{M}$. Then $D_1 \cap \cdots \cap D_n \in \mathcal{M}$ by i). Hence $D_1 \cap \cdots \cap D_n \cap A \neq \emptyset$. □

Proof of Theorem 4.6.1 Let $\mathbb{X} = \prod_{\alpha \in J} \mathbb{X}_\alpha$ where $\forall \alpha \in J$, $\mathbb{X}_\alpha$ is compact. We need to show that $\forall \mathcal{A} \subseteq \mathcal{P}(\mathbb{X})$ satisfying f.i.p., $\bigcap_{A \in \mathcal{A}} Cl(A) \neq \emptyset$. We prove a stronger result. Let $\mathcal{M}$ be a maximal collection satisfying f.i.p. and $\mathcal{A} \subseteq \mathcal{M}$. We show that

$$\bigcap_{M \in \mathcal{M}} Cl(M) \neq \emptyset.$$

Now $\forall \alpha \in J$, consider $\{\pi_\alpha(M) \mid M \in \mathcal{M}\}$ (π_α is the projection map). Since $\mathcal{M}$ satisfies f.i.p., so does $\{\pi_\alpha(M) \mid M \in \mathcal{M}\}$. Since X_α is compact, we have $\bigcap_{M \in \mathcal{M}} Cl(\pi_\alpha(M)) \neq \emptyset$. We choose $x_\alpha \in \bigcap_{M \in \mathcal{M}} Cl(\pi_\alpha(M))$ and define $\underline{x} = (x_\alpha)_{\alpha \in J}$.

Next we show that $\underline{x} \in \bigcap_{M \in \mathcal{M}} Cl(M)$. Let $\beta \in J$ and let U_β be a nbd of x_β in X_β. Consider the typical nbd of $\underline{x}$ of the form

$$\pi_\beta^{-1}(U_\beta) = \Big(\prod_{\alpha \neq \beta} X_\alpha \Big) \times U_\beta .$$

Since $x_\beta \in Cl(\pi_\beta(M))$, $U_\beta \cap \pi_\beta(M) \neq \emptyset \Rightarrow \pi^{-1}(U_\beta) \cap M \neq \emptyset$.

Now we have shown that $\pi_\beta^{-1}(U_\beta) \cap M \neq \emptyset$ for $\forall M \in \mathcal{M}$. So by ii) of Lemma 4.6.3, $\pi_\beta^{-1}(U_\beta) \in \mathcal{M}$. This conclusion holds for all $\beta \in J$ and all nbds U_β of x_β. Hence by i) of Lemma 4.6.3, any finite intersection of $\pi_\beta^{-1}(U_\beta)$ with $\beta \in J$, $U_\beta \in \mathcal{N}_{x_\beta}$ is also in $\mathcal{M}$. But such intersections form a nbd basis of $\underline{x}$. Thus since $\mathcal{M}$ satisfies f.i.p., we deduce that $\forall M \in \mathcal{M}$, $\forall$ nbd U of $\underline{x}$, $U \cap M \neq \emptyset$, which implies $\underline{x} \in Cl(M)$ for all $M \in \mathcal{M}$. This is what we want to prove. □

§4.7 Compactness in Metric Spaces

In this section, we show that in metric spaces, sequentially compactness is equivalent to compactness.

Definition (Sequentially compactness) Let X be a topological space. We say X is *sequentially compact* (s-compact for short) if every sequence in X has a convergent subsequence.

Theorem 4.7.1 A compact C_1 space is s-compact.

Proof. Suppose that $\mathbb{X}$ is C_1 + compact. Let $\{x_n\}$ be a sequence in $\mathbb{X}$. We first notice that $\exists x_0 \in \mathbb{X}$ s.t. any nbd of x_0 contains infinitely many terms of $\{x_n\}$. Actually, if not, then $\forall x \in \mathbb{X}$, $\exists$ a nbd U_x of x s.t. U_x contains only finitely many terms of $\{x_n\}$, which together with the compactness of $\mathbb{X}$ would imply that $\{x_n\}$ has only finitely many terms, a contradiction.

Now since $\mathbb{X}$ is C_1, we can find a $\searrow$ countable nbd basis $\{V_n\}$ of x_0. Since each V_n contains infinitely many terms of $\{x_n\}$, we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ s.t. $x_{n_k} \in V_k$. Then $x_{n_k} \to x_0$. □

Corollary 4.7.2 A compact metric space is s-compact.

Proof. A metric space is C_1. □

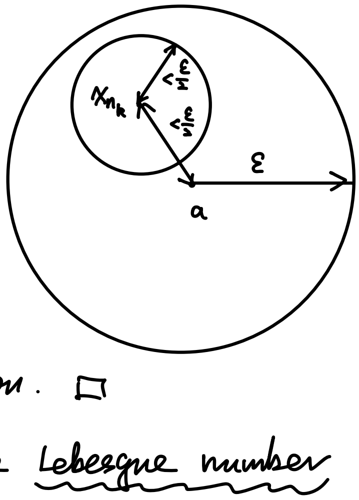
Next we prove the converse conclusion.

Lemma 4.7.3 (Lebesgue) Let $(\mathbb{X}, d)$ be a s-compact metric space. $\mathcal{U}$ is an open covering of $\mathbb{X}$. Then $\exists \delta > 0$, $\forall x \in \mathbb{X}$, $\exists U \in \mathcal{U}$ s.t. $B(x, \delta) \subseteq U$.

Proof. Assume on the contrary that $\forall n \in \mathbb{N}^*$, $\exists x_n \in \mathbb{X}$ s.t. $B(x_n, \frac{1}{n})$ can not be contained in any member of $\mathcal{U}$. Since $\mathbb{X}$ is s-compact, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ s.t. $x_{n_k} \to a \in \mathbb{X}$.

Now since $\mathcal{U}$ is an open covering of $\mathbb{X}$, $\exists \varepsilon > 0$ and $U \in \mathcal{U}$ s.t. $B(a, \varepsilon) \subseteq U$. For k sufficiently large, we have

$$d(x_{n_k}, a) < \frac{\varepsilon}{2}, \quad \frac{1}{n_k} < \frac{\varepsilon}{2}$$



So $\forall y \in B(x_{n_k}, \frac{1}{n_k})$, we have

$$d(y, a) \leq d(y, x_{n_k}) + d(x_{n_k}, a) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.$$

Thus $B(x_{n_k}, \frac{1}{n_k}) \subseteq B(a, \varepsilon) \subseteq U$, which is a contradiction. □

Remark The supreme of δ s.t. Lemma 4.7.3 holds is called the Lebesgue number of the covering $\mathcal{U}$ and is usually denoted by $L(\mathcal{U})$. We note that $L(\mathcal{U})$ rely on the choice of metric d on $\mathbb{X}$.

Definition (Totally Bounded and δ-net)

Let $(\mathbb{X}, d)$ be a metric space, $A \subseteq \mathbb{X}$ and $\delta > 0$. If $\bigcup_{x \in A} B(x, \delta) = \mathbb{X}$, A is called a *δ-net* of $\mathbb{X}$. If $\forall \delta > 0$, $\mathbb{X}$ has a finite δ-net, we say $(\mathbb{X}, d)$ is totally bounded.

Remark It is not hard to see a totally bounded metric space is bounded.

Remark Totally boundedness is not merely a topological property. It depends on the metric. For example, $(-1,1)$ is totally bounded while $\mathbb{R}$ is not. But $(-1,1)$ is homeomorphic to $\mathbb{R}$.

Theorem 4.7.4. Let $(\mathbb{X}, d)$ be a s-compact metric space. Then $(\mathbb{X}, d)$ is totally bounded.

Proof. Assume on the contrary that $\exists \delta_0 > 0$ s.t. $\mathbb{X}$ does not have a δ_0-net. Then we can recursively define a sequence $\{x_n\}$ s.t. $d(x_n, x_m) \geq \delta_0$ for any $m \neq n$ (since finitely many δ_0-ball can never cover $\mathbb{X}$). This sequence does not have any subsequence. □

Question We have just emphasized that totally boundedness depends on the choice of metric. However, s-compact is just a topological property. How can s-compact guarantee total boundedness?

Theorem 4.7.5 A s-compact metric space is compact.

Proof. Let $(\mathbb{X}, d)$ be a s-compact metric space. Let $\mathcal{U}$ be an open covering of $\mathbb{X}$. By Lebesgue's lemma (Lemma 4.7.3.), $\mathcal{U}$ has the Lebesgue number $L(\mathcal{U})$. Let $0 < \delta < L(\mathcal{U})$. By Theorem 4.7.4, $\mathbb{X}$ has a finite δ-net. That is, $\exists x_1, \cdots, x_n \in \mathbb{X}$, s.t. $\mathbb{X} = \bigcup_{i=1}^{n} B(x_i, \delta)$. However, by the definition of the Lebesgue number, $B(x_i, \delta) \subseteq U_i$ for some $U_i \in \mathcal{U}$. Thus $\{U_1, \cdots, U_n\}$ is a finite subcovering of $\mathcal{U}$. □

To summarize, we have obtained

Theorem 4.7.6 In a metric space $\mathbb{X}$, $\mathbb{X}$ is compact iff $\mathbb{X}$ is s-compact. □

Corollary 4.7.7. $A \subseteq \mathbb{R}^n$ is compact $\Longleftrightarrow$ A is bounded and closed.

Proof. $\Longrightarrow$: Since $\mathbb{R}^n$ is Hausdorff, so a compact subset in closed. By Thm 4.7.5 and Thm 4.7.4, A is totally bounded hence is bounded.

$\Longleftarrow$: Since A is bounded, it is contained in some box $[-M, M]^n$, which is compact by Tychonoff's theorem. Then A is a closed subset of a compact space, hence is compact. □

Corollary 4.7.8 Let $\mathbb{X}$ be a compact space. $f: \mathbb{X} \to \mathbb{R}$ is continuous. Then f is bounded. □

Remark In fact, in a metric space,

totally bounded + complete $\Longrightarrow$ compact / s-compact.

We summarize as follows: in a metric space

$$\begin{tikzcd}
& \text{totally bounded + complete} \arrow[dl, Rightarrow, shift right] \arrow[dr, Rightarrow, shift left] & \\
\text{Compact} \arrow[ur, Rightarrow, shift right] \arrow[rr, Leftrightarrow] \arrow[dr, Rightarrow, shift left] & & \text{s-compact} \arrow[ul, Rightarrow, shift left] \arrow[dl, Rightarrow, shift right] \\
& \text{totally bounded} \arrow[ul, Rightarrow, shift left, "/" marking] \arrow[ur, Rightarrow, shift right, "/" marking] &
\end{tikzcd}$$