Topology Assignment 7

Assignment due date: April 21, 2026

1. Please use A4 paper or paper of a similar size to write your assignment; **do not** submit notebooks.
2. The assignment can be completed in either English or Chinese, but English is recommended. There is no need to copy the questions, but please indicate the question numbers.
3. Please ensure that your name and student ID are written on every sheet of paper.
4. Assignments will be graded solely based on submission, so feel free to complete it according to your own abilities and do not plagiarize.

Required Problems

1. Is the quotient space of a connected space still connected? Is the quotient space of a path-connected space still path-connected? Prove or give counterexamples.

2. Prove that a compact Hausdorff space is T_4.

3. Let X and Y be topological spaces. Let $f : X \to Y$ and let

$$G = \{(x, f(x)) \mid x \in X\} \subset X \times Y.$$

The set G is called the graph of f. If Y is compact and G is a closed subset of $X \times Y$, prove that f is continuous.

4. Prove the Bolzano–Weierstrass theorem: every infinite subset of a compact space has a limit point.

5. Show that $\mathbb{Q}$ are not locally compact.

6. Prove that a closed subset of a locally compact space is also locally compact.

Optional Problems

1. This problem can (and must) be solved using Tychonoff's theorem. Let p be a given prime number. For each positive integer n, denote by

$$A_n = \mathbb{Z}/p^n\mathbb{Z}$$

the ring of residue classes modulo p^n. For each n, we have a canonical ring homomorphism

$$i_n : A_{n+1} \to A_n$$
$$x \pmod{p^{n+1}} \mapsto x \pmod{p^n}.$$

Equip each A_n with the discrete topology, and let X be the product space of all A_n with the product topology. Define the set of p-adic integers (which in fact is a ring under componentwise operations) as

$$\mathbb{Z}_p = \{(x_n)_{n\geq 1} \in X \mid i_n(x_{n+1}) = x_n,\ \forall n \geq 1\}.$$

Show that $\mathbb{Z}_p$ is a compact subset of X.

2. Let $(\alpha_i)_{i\in I}$ be a net. If there exist a directed set K and a function $g : K \to I$ such that

i). $i \leq j \implies g(i) \leq g(j)$;

ii). $\forall i \in I$, $\exists k \in K$ such that $g(k) \geq i$,

then $(\beta_k)_{k \in K} := (\alpha_{g(k)})_{k \in K}$ is called a subnet of $(\alpha_i)_{i \in I}$. Prove that X is compact if and only if every net in X has a convergent subnet.