Godement–Jacquet Theory for GL(2)

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References:

[Bu 98] D. Bump. "Automorphic Forms and Representations". Cambridge Studies in Advanced Math. 55. Cambridge University Press. 1998

[GH 11] D. Goldfeld & J. Hundley. "Automorphic Representations and L-Functions for the General Linear Group." Cambridge Studies in Advanced Math. 129 & 130. 2011

§3.1 Global Theory (Cuspidal Case)

$\Phi \in \mathcal{S}(M_2(\mathbb{A}))$, $\pi = \otimes'_v \pi_v \subset L^2_0(G_2, \omega)$. f matrix coefficient (of smooth vectors) of π

To study: $Z(s,\Phi,f) := \int_{GL_2(\mathbb{A})} \Phi(g)\, f(g)\, |\det g|^{s+\frac{1}{2}}\, dg$, $\Re s > \frac{3}{2}$

Rmk: We normalize the (Petersson) inner product as

$$\langle f_1, f_2 \rangle = \int_{GL_2(F)\mathbb{R}_{>0}\backslash GL_2(\mathbb{A})} f_1(g)\, \overline{f_2(g)}\, dg ,$$

which differs from the one defined via "$GL_2(F)\mathbb{A}^\times\backslash GL_2(\mathbb{A})$" by a constant.

If we denote $G_2' := \{ g \in GL_2(\mathbb{A}) \mid |\det g|_{\mathbb{A}} = 1 \}$ then

$GL_2(\mathbb{A}) = \mathbb{R}^\times_{>0} \times G_2'$ & we can identify $GL_2(F)\mathbb{R}_{>0}\backslash GL_2(\mathbb{A}) \simeq GL_2(F)\backslash G_2'$.

Thus $f(g) = \langle R(g) f_1, f_2 \rangle$ for some $f_1, f_2 \in \pi$.

Rmk: The global theory for non-cuspidal spectrum (GL_n) can be reduced to the one for cuspidal spectrum of GL_1. It is interesting to work out the detail.

Def. 1. We define the Fourier transform

$$\hat{\Phi}(X) := \int_{M_2(\mathbb{A})} \Phi(Y)\, \psi(-\mathrm{Tr}(X^T Y))\, dY .$$

We also define left & right actions of $GL_2(\mathbb{A})$ as $h.\Phi.g(X) := \Phi(gXh)$.

Def. 2. For $g \in GL_n$ write $g^\iota = (g^T)^{-1}$. Let $f^\iota(g) := f(g^\iota)$.

Lemma 1. (1) We have Poisson summation formula

$$\sum_{\alpha \in M_2(\mathbb{Q})} \Phi(\alpha) = \sum_{\alpha \in M_2(\mathbb{Q})} \hat{\Phi}(\alpha)$$

(2) $\widehat{g.\Phi.h}(X) = g^\iota.\hat{\Phi}.h^\iota(X) \cdot |\det g|^{-2}\, |\det h|^{-2}$.

Thm. 1. The Godement-Jacquet zeta integral $Z(s,\Phi,\beta)$ has meromorphic continuation to all $s\in\mathbb{C}$ with functional equation

$$Z(s,\Phi,\beta)=Z(1-s,\hat{\Phi},\beta^{\iota}).$$

Rmk: (1) Our convention of Fourier transform differs from [GH11] by a transpose. This is responsible for the appearance of β^{ι} instead of $\check{\beta}$ ($\check{\beta}(g):=\beta(g^{-1})$).

(2) For irreducible π, β^{ι} is a matrix coefficient of the contragredient $\pi^{\vee}$ of π. This a non-trivial fact, whose proof can be found in Theorem 4.2.2. [Bu98].

For $\operatorname{Re} s \gg 1$ we compute (take $\beta(g)=\langle\pi(g)f_1,f_2\rangle$)

$$\begin{aligned}Z(s,\Phi,\beta)&=\int_{GL_2(\mathbb{A})}\Phi(g)\left(\int_{GL_2(F)\backslash G^1} f_1(h_2 g)\overline{f_2(h_2)}\,dh_2\right)|\det g|_{\mathbb{A}}^{s+\frac{1}{2}}\,dg\\
&=\int_{GL_2(\mathbb{A})}\int_{GL_2(F)\backslash G^1}\Phi(h_2^{-1}g)f_1(g)\overline{f_2(h_2)}\,|\det g|_{\mathbb{A}}^{s+\frac{1}{2}}\,dh\,dg\\
&=\int_0^{\infty}\Bigg[\int_{GL_2(F)\backslash G^1}\int_{GL_2(F)\backslash G^1}\Big(\sum_{\xi\in GL_2(F)}\Phi\Big(h_2^{-1}\xi\begin{pmatrix}t&\\&t\end{pmatrix}h_1\Big)\Big)f_1(h_1)\overline{f_2(h_2)}\,dh_1\,dh_2\Bigg]\\
&\qquad\omega(t)\,t^{2s+1}\,d^{\times}t\end{aligned}$$

In the above we change $g=\xi\begin{pmatrix}t&\\&t\end{pmatrix}h_1$ for $\xi\in GL_2(F)$, $t>0$ & $h_1\in GL_2(F)\backslash G^1$.

Write $Z(s,\Phi,\beta)=I_{\infty}(s,\Phi,\beta)+I_0(s,\Phi,\beta)$ with $I_{\infty}=\int_1^{\infty}\cdots$ & $I_0=\int_0^1\cdots$

* Then $I_{\infty}(s,\Phi,\beta)$ is abs. conv. for all $s\in\mathbb{C}$ since it is equal to

$$I_{\infty}(s,\Phi,\beta)=\int_{|\det g|_{\mathbb{A}}\geq 1}\Phi(g)\beta(g)|\det g|_{\mathbb{A}}^{s+\frac{1}{2}}\,dg \ll \int_1^{\infty}\left(\int_{G^1}\Big|\Phi\Big(\begin{pmatrix}t&\\&t\end{pmatrix}h\Big)\Big|dh\right)t^{2\sigma+1}\frac{dt}{t}$$

for $\sigma=\operatorname{Re}(s)\in\mathbb{R}$. But for $t>1$ we have $t^{2\sigma+1}\leq t^{\max(4,2\sigma+1)}$. Hence

$$\begin{aligned}I_{\infty}(s,\Phi,\beta)&\ll\int_0^{\infty}\left(\int_{G^1}\Big|\Phi\Big(\begin{pmatrix}t&\\&t\end{pmatrix}h\Big)\Big|dh\right)t^{\max(4,2\sigma+1)}\frac{dt}{t}\\
&=\int_{GL_2(\mathbb{A})}|\Phi(g)|\cdot|\det g|^{\max(2,\sigma+\frac{1}{2})}\,dg\\
&=\int_{M_2(\mathbb{A})}|\Phi(X)|\cdot|\det X|^{\max(0,\sigma-\frac{3}{2})}\,dX<+\infty.\end{aligned}$$

* To treat $I_0(s,\Phi,\beta)$ we apply the Poisson summation formula to rewrite

$$I_0(s,\Phi,\beta)=I_1+I_2+I_3-I_4-I_5 \quad\text{with}$$

$$I_1=\int_1^{\infty}\Bigg[\int_{GL_2(F)\backslash G^1}\int_{GL_2(F)\backslash G^1}\Big(\sum_{\xi\in GL_2(F)}\hat{\Phi}\Big(h_2^{-1}\xi\begin{pmatrix}t&\\&t\end{pmatrix}h_1\Big)\Big)f_1(h_1^{\iota})\overline{f_2(h_2^{\iota})}\,dh_1\,dh_2\Bigg]t^{3-2s}\frac{dt}{t}$$

$$I_2=\int_1^{\infty}\Bigg[\int_{GL_2(F)\backslash G^1}\int_{GL_2(F)\backslash G^1}\Big(\sum_{\xi\in M_2(F)}\hat{\Phi}\Big(h_2^{-1}\xi\begin{pmatrix}t&\\&t\end{pmatrix}h_1\Big)\Big)f_1(h_1^{\iota})\overline{f_2(h_2^{\iota})}\,dh_1\,dh_2\Bigg]t^{3-2s}\frac{dt}{t}$$

$$I_3 = \int_1^\infty \left[\int_{GL_2(F)\backslash G^1} \int_{GL_2(F)\backslash G^1} \hat{\Phi}\begin{pmatrix} 0&0\\0&0 \end{pmatrix} f_1(h_1) \overline{f_2(h_2)}\, dh_1 dh_2 \right] t^{3-2s} \frac{dt}{t}$$

$$I_4 = \int_0^1 \left[\int_{GL_2(F)\backslash G^1} \int_{GL_2(F)\backslash G^1} \Big(\sum_{\substack{\xi \in M_2(F) \\ \mathrm{rk}(\xi)=1}} \Phi\Big(h_2^{-1} \xi \begin{pmatrix} t & \\ & t \end{pmatrix} h_1\Big) \Big) f_1(h_1) \overline{f_2(h_2)}\, dh_1 dh_2 \right] t^{2s+1} \frac{dt}{t}$$

$$I_5 = \int_0^1 \left[\int_{GL_2(F)\backslash G^1} \int_{GL_2(F)\backslash G^1} \Phi\begin{pmatrix} 0&0\\0&0 \end{pmatrix} f_1(h_1) \overline{f_2(h_2)}\, dh_1 dh_2 \right] t^{2s+1} \frac{dt}{t}$$

Lemma 2. $I_3 = I_5 = 0$

Proof. Easy because $f_1 \perp 1$, $f_2 \perp 1$. □

Lemma 3. $I_2 = I_4 = 0$

Proof. Orbits of $GL_2(F)$ on rank 1 matrices are $GL_2(F)\begin{pmatrix} 0&0\\0&1 \end{pmatrix}\gamma_1$ for $\gamma_1 \in GL_2(F)$

We have $GL_2(F)\begin{pmatrix} 0&0\\0&1 \end{pmatrix}\gamma_1 \simeq P(F)\backslash GL_2(F)$, $\gamma_2^{-1}\begin{pmatrix} 0&0\\0&1 \end{pmatrix}\gamma_1 \leftarrow [\gamma_2]$

where $P = \left\{\begin{pmatrix} * & 0\\ * & 1 \end{pmatrix}\right\} > U = \left\{\begin{pmatrix} 1&0\\ * & 1 \end{pmatrix}\right\}$. $(= N^T)$

For I_2, the inner integral for this orbit over h_2 is

$$\int_{GL_2(F)\backslash G^1} \sum_{\gamma_2 \in P(F)\backslash GL_2(F)} \hat{\Phi}\Big(h_2^{-1}\gamma_2^{-1}\begin{pmatrix} 0&0\\0&1 \end{pmatrix}\begin{pmatrix} t & \\ & t \end{pmatrix}\gamma_1 h_1\Big) \overline{f_2(h_2)}\, dh_2$$

$$= \int_{P(F)\backslash G^1} \hat{\Phi}\Big(h_2^{-1}\begin{pmatrix} 0&0\\0&1 \end{pmatrix}\begin{pmatrix} t & \\ & t \end{pmatrix}\gamma_1 h_1\Big) \overline{f_2(h_2)}\, dh_2$$

$$= \int_{U(\mathbb{A})P(F)\backslash G^1} \hat{\Phi}\Big(h_2^{-1}\begin{pmatrix} 0&0\\0&1 \end{pmatrix}\begin{pmatrix} t & \\ & t \end{pmatrix}\gamma_1 h_1\Big) \Big(\underbrace{\int_{U(F)\backslash U(\mathbb{A})} \overline{f_2(u^t h_2)}\, du}_{= \overline{f_{2,N}}(h_2^t) = 0} \Big)\, dh_2$$

Therefore $I_2 = 0$. Similarly, $I_4 = 0$. □

We obtain $I_1(s, \Phi, \varphi) = I_1$ hence

$$Z(s,\Phi,\varphi) = \int_1^\infty \left[\int_{GL_2(F)\backslash G^1} \int_{GL_2(F)\backslash G^1} \Big(\sum_{\xi \in GL_2(F)} \Phi\Big(h_2^{-1}\xi \begin{pmatrix} t & \\ & t \end{pmatrix} h_1\Big)\Big) f_1(h_1) \overline{f_2(h_2)}\, dh_1 dh_2 \right] t^{2s+1} \frac{dt}{t}$$

$$+ \int_1^\infty \left[\int_{GL_2(F)\backslash G^1} \int_{GL_2(F)\backslash G^1} \Big(\sum_{\xi \in GL_2(F)} \hat{\Phi}\Big(h_2^{-1}\xi \begin{pmatrix} t & \\ & t \end{pmatrix} h_1\Big) f_1(h_1^\iota) \overline{f_2(h_2^\iota)}\, dh_1 dh_2 \right] t^{3-2s} \frac{dt}{t}$$

which is abs. conv. for all $s \in \mathbb{C}$ & invariant under $\Phi \leftrightarrow \hat{\Phi}$, $\varphi \leftrightarrow \varphi^\iota$, $s \leftrightarrow 1-s$.

§3.2 Local Theory

For decomposable $\Phi = \otimes_v' \Phi_v$ & $\varphi = \otimes_v' \varphi_v$ we have for $\mathrm{Re}\, s > \frac{3}{2}$

$$Z(s, \Phi, \varphi) = \prod_v \int_{GL_2(F_v)} \Phi_v(g)\, \varphi_v(g)\, |\det g|_v^{s+\frac{1}{2}}\, dg =: \prod_v Z_v(s, \Phi_v, \varphi_v)$$

We need to distinguish cases for π_v. For brevity, we only treat two cases:

(1) $\pi_v = \pi(\chi_v, \omega_v\chi_v^{-1})$ is a principal series (not necessarily unitary)

(1) [illegible] is a principal series ([illegible]).

(2) $v < \infty$ & π_v is a supercuspidal.

For simplicity of notation, we omit the subscript v from now on, and all (local) representations & vectors are smooth / K-finite.

Lemma 4. (1) The local $Z(s, \Phi, \beta)$ is abs. conv. for $\mathrm{Re}\, s > \frac{1}{2}$.

(2) If π is supercuspidal then $Z(s, \Phi, \beta)$ is abs. conv. for $\mathrm{Re}\, s > -\frac{1}{2}$.

Proof: We only treat (1), leaving (2) as an exercise. (See proof of Prop. 1.)

Let $K < GL_2(F)$ be a maximal compact (connected) subgp, e.g.

$$K = \begin{cases} SU_2(\mathbb{C}) & F = \mathbb{C} \\ SO_2(\mathbb{R}) & F = \mathbb{R} \\ GL_2(\mathcal{O}) & F \text{ non-arch.} \end{cases}$$

Then $\begin{pmatrix} t_1 & u \\ & t_2 \end{pmatrix} \mapsto \int_K \left| \Phi\left(\begin{pmatrix} t_1 & u \\ & t_2 \end{pmatrix} k\right) \underbrace{\beta\left(\begin{pmatrix} t_1 & u \\ & t_2 \end{pmatrix} k\right)}_{\leq 1} \right| dk$ is majorized by a positive $\phi \in S(F^3)$. Thus $Z(s, \Phi, \beta)$ is majorized via Iwasawa decomposition as

$$\int_{F^\times \times F^\times \times F} \phi(t_1, t_2, u) \, |t_1 t_2|^{\mathrm{Re}\, s + \frac{1}{2}} \frac{d^\times t_1}{|t_1|} d^\times t_2 \, du$$

which is convergent for $\mathrm{Re}\, s > \frac{1}{2}$. $\square$

Rmk. Ex. 11.12 of [GH11] claims the abs. conv. for $\mathrm{Re}\, s > -\frac{1}{2}$, which looks too good to believe. I think that $\mathrm{Re}\, s > 0$ is plausible if one takes into account the decay of matrix coefficients.

Thm. 2. The local $Z(s, \Phi, \beta)$ has meromorphic continuation to $s \in \mathbb{C}$ with

$$Z(1-s, \hat{\Phi}, \beta^\vee) = \gamma(s, \pi, \psi) \, Z(s, \Phi, \beta)$$

for some $\gamma(s, \pi, \psi) \in \mathbb{C}$ independent of Φ & β.

* Principal Series

$\pi = \pi(\chi_1, \chi_2)$ for (quasi-) characters χ_1, χ_2 of $F^\times$ s.t. $\chi_1 \chi_2 = \omega$. It is realized in

$$\mathcal{B}(\chi_1, \chi_2) = \mathrm{Ind}_{B(F)}^{GL_2(F)} (\chi_1, \chi_2)$$
$$= \left\{ f \in C^\infty(GL_2(F)) \,\middle|\, f\left(\begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix} g\right) = \left|\frac{a_1}{a_2}\right|^{\frac{1}{2}} \chi_1(a_1) \chi_2(a_2) f(g) \right\}$$

The contragredient $\tilde{\pi} = \pi(\chi_1^{-1}, \chi_2^{-1})$, so that matrix coefficient β can be written as $\beta(g) = \langle \pi(g) f, \tilde{f} \rangle = \int_K f(kg) \tilde{f}(k) \, dk$

for $f \in \mathcal{B}(\chi_1, \chi_2)$, $\tilde{f} \in \mathcal{B}(\chi_1^{-1}, \chi_2^{-1})$. Hence for $\mathrm{Re}\, s \gg 1$

$$Z(s, \Phi, \beta) = \int_{GL_2(F)} \Phi(g) \left[\int_K f(xg) \tilde{f}(x)\, dx\right] |\det g|^{s+\frac{1}{2}} dg$$

$$= \int_{GL_2(F)} \left[\int_K \Phi(x^{-1}g)\, f(g)\, \tilde{f}(x)\, dx\right] |\det g|^{s+\frac{1}{2}} dg$$

$$= \int_{F^\times \times F^\times \times F} \left\{\int_K \int_K \Phi\left(x^{-1}\begin{pmatrix} t_1 & u \\ & t_2 \end{pmatrix} x'\right) f(x') \tilde{f}(x)\, dx\, dx'\right\} \chi_1(t_1) |t_1|^s \chi_2(t_2) |t_2|^s\, du\, d^\times t_1\, d^\times t_2$$

Note that (at least) for K-finite f & $\tilde{f}$ the function

$$X \mapsto \int_K \int_K \Phi(x^{-1} X x')\, f(x') \tilde{f}(x)\, dx\, dx'$$

is still Schwartz-Bruhat. Hence the function

$$f_\Phi(t_1, t_2) := \int_F \left[\int_K \int_K \Phi\left(x^{-1}\begin{pmatrix} t_1 & u \\ & t_2 \end{pmatrix} x'\right) f(x') \tilde{f}(x)\, dx\, dx'\right] du$$

is still Schwartz-Bruhat. Therefore $Z(s, \Phi, \beta)$ is a 2-dim'l Tate's integral & admits meromorphic continuation, F.E. etc.

Lemma 5. For $\Phi \in S(M_2(F))$, let $T_\Phi(t_1, t_2) = \int_F \Phi\begin{pmatrix} t_1 & u \\ & t_2 \end{pmatrix} du$. Then

$$\widehat{T_\Phi}(t_1, t_2) = \int_F \widehat{\Phi}\begin{pmatrix} t_1 & \\ u & t_2 \end{pmatrix} du.$$

Proof: $\widehat{T_\Phi}(t_1, t_2) = \int_{F^2} \left(\int_F \Phi\begin{pmatrix} a_1 & u \\ & a_2 \end{pmatrix} du\right) \psi(-a_1 t_1 - a_2 t_2)\, da_1\, da_2$ &

$$\int_F \widehat{\Phi}\begin{pmatrix} t_1 & \\ u & t_2 \end{pmatrix} du = \int_F \left[\int_{M_2(F)} \Phi\begin{pmatrix} a_1 & v_1 \\ v_2 & a_2 \end{pmatrix} \psi\Bigg(-\underbrace{\mathrm{Tr}\left(\begin{pmatrix} a_1 & v_2 \\ v_1 & a_2 \end{pmatrix}\begin{pmatrix} t_1 & \\ u & t_2 \end{pmatrix}\right)}_{= a_1 t_1 + v_2 u + a_2 t_2}\Bigg)\, da_1\, da_2\, dv_1\, dv_2\right] du$$

$$= \int_{F^3} \Phi\begin{pmatrix} a_1 & v_1 \\ & a_2 \end{pmatrix} \psi(-a_1 t_1 - a_2 t_2)\, da_1\, da_2 = \widehat{T_\Phi}(t_1, t_2) \qquad \square$$

From Lemma 5. we get

$$\widehat{f_\Phi}(t_1, t_2) = \int_K \int_K \widehat{T_{x\Phi x'}}(t_1, t_2)\, f(x') \tilde{f}(x)\, dx\, dx'$$

$$= \int_K \int_K \left(\int_F \widehat{\Phi}\left(x^T \begin{pmatrix} t_1 & \\ u & t_2 \end{pmatrix} (x')^T\right) du\right) f(x') \tilde{f}(x)\, dx\, dx'.$$

$$\int_{F^\times \times F^\times} \widehat{f_\Phi}(t_1, t_2)\, \chi_1(t_1)^{-1} \chi_2(t_2)^{-1} |t_1|^{1-s} |t_2|^{1-s}\, d^\times t_1\, d^\times t_2 \qquad (\mathrm{Re}\, s \ll -1)$$

$$= \int_{F^\times \times F^\times \times F} \left[\int_K \int_K \left(\int_F \widehat{\Phi}\left(\left(\begin{pmatrix} t_1 & u \\ & t_2 \end{pmatrix} x\right)^T x'\right) du\right) f((x')^T) \tilde{f}\left(\begin{pmatrix} t_1 & u \\ & t_2 \end{pmatrix} x\right) dx\, dx'\right]$$

$$|t_1 t_2|^{\frac{3}{2}-s} \cdot |t_1|^{-1}\, d^\times t_1\, d^\times t_2$$

$$= \int_K \int_{GL_2(F)} \widehat{\Phi}(g\, x')\, f((x')^T)\, \tilde{f}(g^T)\, |\det g|^{\frac{3}{2}-s}\, dg$$

$$= \int_{GL_2(F)} \widehat{\Phi}(g) \left[\int_K f((x')^T) \tilde{f}(((x')^{-1} g)^T)\, dx'\right] |\det g|^{\frac{3}{2}-s}\, dg$$

$$= \int_{GL_2(F)} \widehat{\Phi}(g) \left[\int_K f(x' \cdot g') \tilde{f}(x')\, dx'\right] |\det g|^{\frac{3}{2}-s}\, dg$$

$$= Z(1-s, \widehat{\Phi}, \beta')$$

This proves the local F.E. with

$$\gamma(s,\pi,\psi) = \gamma(s,\chi_1,\psi)\,\gamma(s,\chi_2,\psi).$$

* Supercuspidals

Recall: ① β has compact support modulo $F^\times$ (center)

② $\int_F \beta\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}g\right)dx = \int_F \beta\left(g\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right)dx = 0$

$\int_N \pi(n)(\pi(n)v - v)\,dn = 0$

Prop. 1. For supercuspidal π, $Z(s,\Phi,\beta)$ are entire in s.

Proof: $Z(s,\Phi,\beta) = \int_{F^\times\backslash GL_2(F)}\left(\int_{F^\times}\Phi\left(\begin{pmatrix}t & \\ & t\end{pmatrix}g\right)\omega(t)|t|^{2s+1}d^\times t\right)\beta(g)\,|\det g|^{s+\frac{1}{2}}dg$

The outer integral is non-vanishing over a compact subset $S \subset GL_2(F)$ representative for $F^\times\backslash GL_2(F)$, on which $\beta(g)\,|\det g|^{s+\frac{1}{2}}$ is bounded for s lying in finite vertical regions. The inner integral is Tate's integral & has meromorphic cont. to all $s\in\mathbb{C}$ with simple pole at $s_0\in i\mathbb{R}$ only if ω is unramified. The residue is equal to $\Phi(0)$ indep. of g.

But $\int_{F^\times\backslash GL_2(F)}\beta(g)\,|\det g|^{s+\frac{1}{2}}dg = 0$ by first integrating over N. □

Def. 3. Let $S_0(M_2(F))$ be the subspace of $\Phi\in S(M_2(F))$ s.t. both Φ and $\hat{\Phi}$ have support contained in $GL_2(F)$.

Rmk: (1) The fact $S_0(M_2(F)) \neq \{0\}$ can be seen in the proof of Lemma 8 below.

(2) If $\Phi\in S_0(M_2(F))$ then $\Phi(g)=0$ for $|\det g|<\varepsilon$ for some $\varepsilon>0$. This is because $x\mapsto|\det(x)|$ is continuous. Hence $|\det(\mathrm{Supp}\,\Phi)|$ is compact not containing 0. Consequently $\mathrm{Supp}\,\Phi$ is a compact subset of $GL_2(F)$.

Lemma 6. Let $(\tilde{\pi},\tilde{V})$ be the contragredient of (π,V) and let $\langle,\rangle: V\times\tilde{V}\to\mathbb{C}$ be the $GL_2(F)$-invariant pairing. Take $\Phi\in S(M_2(F))$ and $\Psi\in S_0(M_2(F))$, then we have for $v\in V$ & $\tilde{v}\in\tilde{V}$

$$\int_{GL_2(F)}\int_{GL_2(F)}\Phi(g)\,\hat{\Psi}(h)\,\langle\pi(g)v,\tilde{\pi}(h^{T})\tilde{v}\rangle\,|\det g|^{s+\frac{1}{2}}\,|\det h|^{\frac{3}{2}-s}\,dg\,dh$$

$$= \int_{GL_2(F)} \int_{GL_2(F)} \hat{\Phi}(g)\, \Psi(h) \langle \pi(g^{-1}).v, \tilde{\pi}(h^{-1})\tilde{v} \rangle |\det g|^{\frac{3}{2}-s} |\det h|^{s+\frac{1}{2}}\, dg\, dh$$

where: – the LHS is abs. conv. for $\mathrm{Re}\, s > -\frac{1}{2}$

– the RHS is abs. conv. for $\mathrm{Re}\, s < \frac{3}{2}$

Proof: ① Since $\Psi \in S_0(M_2(F))$, the integrals over h on both sides can be replaced by finite sums & causes no problem of convergence for any $s \in \mathbb{C}$. The abs. conv. regions then follow from Lemma 4. (2).

② Let $-\frac{1}{2} < \mathrm{Re}\, s < \frac{3}{2}$. A change of variables $g \mapsto h^T g$ gives

$$LHS = \int_{GL_2(F)} \langle \pi(g).v, \tilde{v} \rangle \left(\int_{M_2(F)} \Phi(h^T g)\, \hat{\Psi}(h)\, dh \right) |\det g|^{s+\frac{1}{2}} dg$$

where dh is the additive Haar measure on $M_2(F)$ (abus of notation).

Similarly, the same change of variables gives

$$RHS = \int_{GL_2(F)} \langle \pi(g^{-1}).v, \tilde{v} \rangle \left(\int_{M_2(F)} \hat{\Phi}(h^T g)\, \Psi(h)\, dh \right) |\det g|^{\frac{3}{2}-s} dg$$

$$= \int_{GL_2(F)} \langle \pi(g).v, \tilde{v} \rangle \left(\int_{M_2(F)} \hat{\Phi}(h^T g^{-1})\, \Psi(h)\, dh \right) |\det g|^{s-\frac{3}{2}} dg.$$

Note that the Plancherel formula implies

$$\int_{M_2(F)} \Phi(h^T g)\, \hat{\Psi}(h)\, dh = \int_{M_2(F)} g\Phi(h^T)\, \hat{\Psi}(h)\, dh$$

$$= \int_{M_2(F)} \widehat{g\Phi}(h^T)\, \Psi(h)\, dh = |\det g|^{-2} \int_{M_2(F)} \hat{\Phi}(h^T g^{-1})\, \Psi(h)\, dh$$

and the desired equality follows.

Rmk: Let $\pi^{\iota}(g) := \pi(g^{-1})$. Define operator valued zeta integral

$$Z(s, \Phi, \pi) := \int_{GL_2(F)} \Phi(g) |\det g|^{s+\frac{1}{2}} \pi(g)\, dg : V \to V$$

If the support of Φ is compact in $GL_2(F)$, then the above integral is in fact a finite sum & converges abs. for any $s \in \mathbb{C}$. Lemma 6. is equivalent to

$$Z(1-s, \hat{\Psi}, \pi^{\iota}) \circ Z(s, \Phi, \pi) = Z(s, \Psi, \pi) \circ Z(1-s, \hat{\Phi}, \pi^{\iota}).$$

To proceed we postpone the proofs of the following two technical lemmas.

Lemma 7. Given $v \in V$, $\tilde{v} \in \tilde{V}$ and $s \in \mathbb{C}$, there exists $\Phi \in S_0(M_2(F))$ s.t.

$$Z(s, \Phi, \pi).w = \langle w, \tilde{v} \rangle v \quad \text{for all } w \in V.$$

Lemma 8. Given $v \in V$, define the subset

$$U_v := \left\{ u \in V \;\middle|\; \begin{array}{l} \exists\, \Phi \in S_0(M_n(F)),\ c \neq 0,\ n \in \mathbb{Z} \text{ s.t.} \\ Z(s, \Phi, \pi).v = c\, q^{-ns} u \quad \forall s \in \mathbb{C} \end{array} \right\}$$

Then U_v spans V for all $v \in V$.

We can now deduce a preliminary local F.E.

Lemma 9. For all $s \in \mathbb{C}$ there $\exists!\ \gamma(s) \in \mathbb{C}$ s.t.

$$Z(1-s, \hat{\Phi}, \pi^\vee) = \gamma(s)\, Z(s, \Phi, \pi) \quad \text{for all } \Phi \in S_0(M_n(F)).$$

Proof: ① By Lemma 7, for any $s \in \mathbb{C}$, $v \in V$ we can find $\Phi \in S_0(M_n(F))$ s.t. $Z(s, \Phi, \pi).v = v$. Then an operator $\gamma(s): V \to V$ which satisfies the desired equation must be given by $\gamma(s).v = Z(1-s, \hat{\Phi}, \pi^\vee).v$.

To see such $\gamma(s)$ is well-defined, we must show that if $\Phi_1, \Phi_2 \in S_0(M_n(F))$ s.t. $Z(s, \Phi_1, \pi).v = Z(s, \Phi_2, \pi).v = v$ then

$$Z(1-s, \hat{\Phi}_1, \pi^\vee).v = Z(1-s, \hat{\Phi}_2, \pi^\vee).v = v.$$

To this end, let $w = Z(1-s, \hat{\Phi}_1, \pi^\vee).v - Z(1-s, \hat{\Phi}_2, \pi^\vee).v = Z(1-s, \hat{\Phi}_1 - \hat{\Phi}_2, \pi^\vee).v$

Choose $\Psi \in S_0(M_n(F))$ s.t. $Z(s, \Psi, \pi).w = w$ by Lemma 7. Then

$$\begin{aligned} &Z(s, \Psi, \pi).Z(1-s, \hat{\Phi}_1 - \hat{\Phi}_2, \pi^\vee).v = w \\ &\quad \| \text{ Lemma 6.} \\ &Z(1-s, \hat{\Psi}, \pi^\vee).Z(s, \Phi_1 - \Phi_2, \pi).v = 0 \end{aligned}$$

② Similar method to ① can show that $\gamma(s)$ is linear. We leave the details as an exercise.

③ Let $h \in GL_n(F)$ we have $\widehat{\Phi.h} = (\hat{\Phi}.h^{-1}) \cdot |\det h|^{-n}$, and

$$\Phi \in S_0(M_n(F)) \Rightarrow h.\Phi \in S_0(M_n(F)). \quad \text{Hence}$$

$$\left.\begin{aligned} Z(s, \Phi h, \pi) &= |\det h|^{-s-\frac{1}{2}}\, \pi(h^{-1}) \circ Z(s, \Phi, \pi) \quad (1) \\ Z(1-s, \widehat{\Phi.h}, \pi^\vee) &= |\det h|^{-s+\frac{1}{2}}\, \pi(h^{-1}) \circ Z(1-s, \hat{\Phi}, \pi^\vee) \end{aligned}\right\} \Rightarrow$$

$$\pi(h^{-1}) \circ \gamma(s) \circ Z(s, \Phi, \pi) = \gamma(s) \circ \pi(h^{-1}) \circ Z(s, \Phi, \pi) \quad \text{for all } \Phi \in S_0(M_n(F))$$

$$\Rightarrow \pi(h^{-1}) \circ \gamma(s) = \gamma(s) \circ \pi(h^{-1}) \quad \text{by Lemma 7 again.}$$

By irreducibility of π, the operator $\gamma(s)$ must be a scalar. $\square$

For general $\Phi \in S(M_2(F))$, $v \in V$ & $\Psi \in S_0(M_2(F))$ we have s.t.

$$Z(s,\Psi,\pi)\circ Z(1-s,\hat{\Phi},\pi^\vee).v = Z(1-s,\hat{\Phi},\pi^\vee)\circ Z(s,\Phi,\pi).v = \gamma(s)\, Z(s,\Psi,\pi)\circ Z(s,\Phi,\pi).v$$

$$\Rightarrow \quad Z(s,\Psi,\pi).\big(Z(1-s,\hat{\Phi},\pi^\vee).v - \gamma(s)\, Z(s,\Phi,\pi).v\big) = 0$$

$$\| \text{ Choose } \Psi \text{ by Lemma 7}$$

$$Z(1-s,\hat{\Phi},\pi^\vee).v - \gamma(s)\, Z(s,\Phi,\pi).v\ .$$

Hence we get $Z(1-s,\hat{\Phi},\pi^\vee) = \gamma(s)\, Z(s,\Phi,\pi)$, proving Thm. 2 in this case.

Exercise: Show that $\gamma(s) = \gamma(s,\pi,\psi)$ is a monomial rational function in q^{-s} if q is the cardinality of the residual field of F.

* Technical Lemmas

Proof of Lemma 8. Take $\hat{v} \in \hat{V}$ with $\langle v, \hat{v}\rangle = 1$ & $f(g) := \langle \pi(g).v, \hat{v}\rangle$. Define

$$\Phi(g) = \begin{cases} \overline{f(g)} & \text{if } g \in GL_2(F) \text{ \& } |\det g| = 1 \\ 0 & \text{otherwise} \end{cases} \qquad (*)$$

(1) We first show $\hat{\Phi} \in S_0(M_2(F))$, i.e., $\hat{\Phi}$ has support in $GL_2(F)$.

Claim: $\int_F \Phi\left(g_1 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g_2\right) dx = 0$ for $\forall g_1, g_2 \in GL_2(F)$ $\quad (2)$

In fact, Claim holds trivially if $|\det(g_1)\cdot\det(g_2)| \neq 1$ by considering the support.

Assume $|\det(g_1)\cdot\det(g_2)| = 1$. Set $v' = \pi(g_2).v$ & $\hat{v}' = \hat{\pi}(g_1^{-1}).\hat{v}$. Then (2) becomes

$$\int_F \left\langle \pi\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}.v', \hat{v}'\right\rangle dx = 0 \qquad (3)$$

Since the Jacquet module of π is 0, v' is a finite linear combination of vectors like $\pi(n_0).v_1 - v_1$. Hence (3) holds, proving Claim.

Consider $m \in M_2(F) - GL_2(F)$. Then we can find $g_1, g_2 \in GL_2(F)$ & $a \in \{0,1\}$ s.t. $m = g_1 \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} g_2$. Then

$$\hat{\Phi}(m) = \int_{M_2(F)} \Phi(X)\, \psi(-\mathrm{Tr}(X^T m))\, dX$$

$$= |\det g_1|^2 |\det g_2|^2 \int_{M_2(F)} \Phi(g_1^T X g_2^T)\, \psi\left(-\mathrm{Tr}\left(X^T \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right)\right) dX$$

$$= |\det g_1|^2 \cdot |\det g_2|^2 \int_{GL_2(F)} \Phi(g_1^T g g_2^T)\, \psi\left(-\mathrm{Tr}\left(g^T \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right)\right) \cdot |\det g|^2\, dg$$

We may use Iwasawa decomposition $GL_2(F) = KAN$ and the Haar measure

$$\hat{\Phi}(m) = |\det g_1|^2 \cdot |\det g_2|^2 \cdot \int_{GL_2(F)/N(F)} \psi\left(-\mathrm{Tr}\left(g^T \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right)\right) \cdot \left(\int_F \Phi\left(g_1^T g \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g_2^T\right) dx\right) dg$$

because $\mathrm{Tr}\left(\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} g^T \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right) = \mathrm{Tr}\left(g^T \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}\right) = \mathrm{Tr}\left(g^T \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right)$.

The inner integral vanishes by Claim. Thus $\mathrm{supp}\,\hat{\Phi} \subset GL_2(F)$.

(2) We have $Z(s, \Phi, \beta) = \int_{G'} |\beta(g)|^2 dg \neq 0$ is a constant independent of s $\Rightarrow 0 \neq u = Z(s, \Phi, \pi).v \in U_v$. Then (1) shows that $\pi(h^{-1}).u \in U_v$ for any $h \in GL_2(F)$. Hence the span of U_v is V by irreducibility.

Recall: Schur orthogonality for unitary supercuspidal representations

Prop. 2. For all $v, w \in V$ & $\tilde{v}, \tilde{w} \in \tilde{V}$ we have

$$\int_{F^\times \backslash GL_2(F)} \langle \pi(g).v, \tilde{v} \rangle \langle w, \tilde{\pi}(g).\tilde{w} \rangle dg = c \cdot \langle v, \tilde{w} \rangle \cdot \langle w, \tilde{v} \rangle$$

for some constant $0 \neq c \in \mathbb{C}$ (formal degree of π).

__Sketch of Proof__: Imitate the usual Schur's lemma. Show that the bilinear form

$$B_{w,\tilde{v}} : V \times \tilde{V} \to \mathbb{C}, \ (v, \tilde{w}) \mapsto \int_{F^\times \backslash GL_2(F)} \langle \pi(g).v, \tilde{v} \rangle \langle w, \tilde{\pi}(g)\tilde{w} \rangle dg$$

is $GL_2(F)$-invariant, hence is a constant multiple of the canonical pairing. Then show that the constant $c(w, \tilde{v})$ is an invariant bilinear form. $\square$

__Proof of Lemma 7.__ First note that we only need to treat the case $s = -\frac{1}{2}$.

The define the function

$$\Phi(g) = \begin{cases} \langle v, \tilde{\pi}(g).\tilde{v} \rangle & \text{if } |\det g| \in \{1, q\} \\ 0 & \text{otherwise} \end{cases}$$

We have $\Phi \in S_0(M_2(F))$ since it is just the sum of two functions defined by (*). For $w \in V$, $\tilde{w} \in \tilde{V}$ we have

$$\langle Z(-\tfrac{1}{2}, \Phi, \pi).w, \tilde{w} \rangle = \left\langle \int_{|\det g| \in \{1,q\}} \Phi(g)\, \pi(g).w \, dg, \tilde{w} \right\rangle$$

$$= \int_{|\det g| \in \{1,q\}} \langle v, \tilde{\pi}(g)\tilde{v} \rangle \langle \pi(g)w, \tilde{w} \rangle dg$$

But $\{g \in GL_2(F) \mid |\det g| \in \{1, q\}\}$ is a fundamental domain for $\varpi^{2\mathbb{Z}} \backslash GL_2(F)$

Thus up to a constant we get

$$\langle Z(-\tfrac{1}{2}, \Phi, \pi).w, \tilde{w} \rangle = \int_{F^\times \backslash GL_2(F)} \langle \pi(g) w, \tilde{w} \rangle \langle v, \tilde{\pi}(g) \tilde{v} \rangle \, dg$$

$$\overset{\text{Prop. 2}}{=} c \cdot \langle w, \tilde{v} \rangle \cdot \langle v, \tilde{w} \rangle .$$

Dividing Φ by c $\Rightarrow$ $Z(-\tfrac{1}{2}, \Phi, \pi).w = \langle w, \tilde{v} \rangle v$ for all $w \in V$. $\square$