Homework 2

(due Monday, March 29, 2021)

1. Show, using partial summation or otherwise, that

   a) $\sum\limits_{n\le x} \frac{d(n)}{n} = \frac{1}{2}\log^2 x + 2\gamma \log x + O(1)$;

   b) $\sum\limits_{2\le n\le x} \frac{d(n)}{\log n} = x + 2\gamma \frac{x}{\log x} + O(\frac{x}{\log^2 x})$.

2. Show that

$$\sum_{n\le x} \frac{\varphi(n)}{n} \sim \frac{x}{\zeta(2)} > \frac{x}{2}.$$

3. a) Show that

$$\frac{1}{\varphi(n)} = \frac{1}{n}\sum_{d|n} \frac{\mu^2(d)}{\varphi(d)}.$$

   b) Deduce that

$$\sum_{n\le x} \frac{1}{\varphi(n)} = A\log x + O(1), \quad \text{with } A = {\sum}' \frac{1}{n\varphi(n)} \approx 1.93,$$

   where the accent designate summation over all square-free integers.

4. Show that $d(mn) \le d(m)d(n)$ for all $m, n > 0$. Then show that

$$\sum_{n\le x} d^2(n) \ll x\log^3 x$$

   by writing

$$\sum_{n\le x} d^2(n) = \sum_{n\le x} d(n) \sum_{\ell|n} 1$$

   and interchanging summations.

5. Let $\varphi_1(n) = n\sum_{d|n} |\mu(d)|/d$.

   (a) Prove that φ_1 is multiplicative and that $\varphi_1(n) = n\prod_{p|n}(1+p^{-1})$.

   (b) Prove that

$$\varphi_1(n) = \sum_{d^2|n} \mu(d)\sigma(n/d^2).$$

   (c) Prove that

$$\sum_{n\le x} \varphi_1(n) = \frac{\zeta(2)}{2\zeta(4)}x^2 + O(x\log x).$$