CHAPTER 3

The order and average order of magnitude of arithmetic functions

3.1. The symbols "O", "$\ll$", "$\asymp$", "o" and "$\sim$"

Some convenient notations have been introduced for use during the study of integers. Let $g(x)$ be defined and positive for all x in some unbounded set S of positive numbers (which will usually be either the set of positive integers or the set of positive real numbers, but might for example be the set of primes). Then if $f(x)$ is defined on S, and if there is a constant C such that

$$\frac{|f(x)|}{g(x)} < C$$

for all sufficiently large $x \in S$, then we write either $f(x) = O(g(x))$ or $f(x) \ll g(x)$. If there are constants $0 < c < C < \infty$ such that

$$c < \frac{|f(x)|}{g(x)} < C$$

for all sufficiently large $x \in S$, then we write $f(x) \asymp g(x)$. If

$$\lim_{\substack{x \to \infty \\ x \in S}} \frac{f(x)}{g(x)} = 0,$$

we write $f(x) = o(g(x))$, and if

$$\lim_{\substack{x \to \infty \\ x \in S}} \frac{f(x)}{g(x)} = 1,$$

we write $f(x) \sim g(x)$, and say that $f(x)$ is asymptotically equal to $g(x)$.

Example 3.1. For $x \in \mathbb{R}$, we have

$$\begin{aligned}
\sin x &\ll x, \\
\sin x &= O(1), \\
2 + \sin x &\asymp 1, \\
\sqrt{x} &= o(x), \\
x^k &= o(e^x), \quad \text{for every constant } k, \\
\log^k x &= o(x^\alpha), \quad \text{for every pair of constants } k \text{ and } \alpha > 0, \\
[x] &\sim x.
\end{aligned}$$

3.2. The Euler–Maclaurin formula and the partial summation formula

THEOREM 3.2 (Euler–Maclaurin formula). *Let $a < b$ and $a, b \in \mathbb{Z}$. Let $f : [a, b] \to \mathbb{C}$. If f is of class $\mathcal{C}^1$ on $[a, b]$. Then we have*

$$\sum_{a<n\leq b} f(n) = \int_a^b (f(x) + \psi_1(x)f'(x))\mathrm{d}x + \frac{1}{2}(f(b) - f(a)),$$

where $\psi_1(x) = x - [x] - 1/2$ is the saw function.

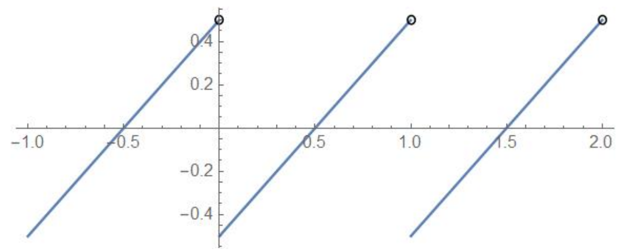

FIGURE 1. The saw function

PROOF. Let $n \in \mathbb{Z}$ such that $a \leq n < b$. By integration by parts, we have

$$\begin{aligned}\int_n^{n+1} \psi_1(x)f'(x)\mathrm{d}x &= \int_n^{n+1} (x - n - 1/2)\mathrm{d}f(x)\\ &= (x - n - 1/2)f(x)\Big|_n^{n+1} - \int_n^{n+1} f(x)\mathrm{d}x\\ &= \frac{1}{2}(f(n+1) + f(n)) - \int_n^{n+1} f(x)\mathrm{d}x.\end{aligned}$$

Hence

$$\begin{aligned}\int_a^b \psi_1(x)f'(x)\mathrm{d}x &= \sum_{n=a}^{b-1} \int_n^{n+1} \psi_1(x)f'(x)\mathrm{d}x\\ &= \frac{1}{2}(f(b) + f(a)) + \sum_{n=a+1}^{b-1} f(n) - \int_a^b f(x)\mathrm{d}x.\end{aligned}$$

So we obtain

$$\sum_{a<n\leq b} f(n) = \int_a^b (f(x) + \psi_1(x)f'(x))\mathrm{d}x + \frac{1}{2}(f(b) - f(a)),$$

as claimed. □

THEOREM 3.3. *We have*

$$\sum_{n\leq x} \log n = x \log x - x + O(\log x).$$

PROOF. By Theorem 3.2 we have

$$\sum_{n\le x}\log n=\int_1^x\left(\log u+\psi_1(u)\frac{1}{u}\right)\mathrm{d}u+O(\log x).$$

Since $|\psi_1(u)|\le 1$, we have

$$\int_1^x\psi_1(u)\frac{1}{u}\mathrm{d}u\ll\int_1^x\frac{1}{u}\mathrm{d}u=O\left(\log x\right).$$

Note that

$$\int_1^x\log u\ \mathrm{d}u=(u\log u-u)\Big|_1^x=x\log x-x+1.$$

This completes the proof. □

THEOREM 3.4. *We have*

$$\sum_{n\le x}\frac{1}{n}=\log x+\gamma+O\left(\frac{1}{x}\right),$$

where γ is the Euler constant.

PROOF. By Theorem 3.2 we have

$$\sum_{n\le x}\frac{1}{n}=1+\int_1^x\left(\frac{1}{u}-\psi_1(u)\frac{1}{u^2}\right)\mathrm{d}u-\frac{1}{2}+O\left(\frac{1}{x}\right).$$

Let $\gamma=\frac{1}{2}-\int_1^\infty\frac{\psi(u)}{u^2}\mathrm{d}u$. Since $|\psi_1(u)|\le 1$, we have

$$\frac{1}{2}-\int_1^x\psi_1(u)\frac{1}{u^2}\mathrm{d}u=\gamma+\int_x^\infty\psi_1(u)\frac{1}{u^2}\mathrm{d}u=\gamma+O\left(\int_x^\infty\frac{1}{u^2}\mathrm{d}u\right)=\gamma+O\left(\frac{1}{x}\right).$$

Note that

$$\int_1^x\frac{1}{u}\mathrm{d}u=\log u\Big|_1^x=\log x.$$

This completes the proof. □

THEOREM 3.5 (partial summation). *Suppose that $\lambda_1,\lambda_2,\cdots$ is a nondecreasing sequence of real numbers with limit infinity, that $c_1,c_2,\cdots$ is an arbitrary sequence of real or complex numbers, and that $f(x)$ has a continuous derivative for $x\ge\lambda_1$. Put*

$$C(x)=\sum_{\lambda_n\le x}c_n,$$

where the summation is over all n for which $\lambda_n\le x$. Then for $x\ge\lambda_1$, we have

$$\sum_{\lambda_n\le x}c_nf(\lambda_n)=C(x)f(x)-\int_{\lambda_1}^x C(t)f'(t)dt.$$

PROOF. Let

$$g(\lambda_n,t)=\begin{cases}0, & \text{if } \lambda_1\le t<\lambda_n,\\ 1, & \text{if } \lambda_n\le t\le x.\end{cases}$$

Since $C(x)=\sum_{\lambda_n\le x} c_n$, we have

$$\begin{aligned}
C(x)f(x)-\sum_{\lambda_n\le x} c_n f(\lambda_n) &= \sum_{\lambda_n\le x} c_n(f(x)-f(\lambda_n)) = \sum_{\lambda_n\le x} c_n\int_{\lambda_n}^x f'(t)dt \\
&= \sum_{\lambda_n\le x} c_n\int_{\lambda_1}^x g(\lambda_n,t)f'(t)dt \\
&= \int_{\lambda_1}^x \sum_{\lambda_n\le x} c_n g(\lambda_n,t)f'(t)dt \\
&= \int_{\lambda_1}^x \sum_{\lambda_n\le t} c_n f'(t)dt = \int_{\lambda_1}^x C(t)f'(t)dt.
\end{aligned}$$

Hence we have

$$\sum_{\lambda_n\le x} c_n f(\lambda_n) = C(x)f(x)-\int_{\lambda_1}^x C(t)f'(t)dt. \qquad \square$$

Take $\lambda_n=n$. Then we get

$$\sum_{n\le x} a_n f(n) = S(x)f(x)-\int_1^x S(u)f'(u)\mathrm{d}u,$$

where $S(u)=\sum_{n\le u} a_n$.

3.3. The order of magnitude of τ, σ and φ

In this section we will give the true order of magnitude of τ, φ and σ.

THEOREM 3.6. *Let $\tau(n)$ be the divisor function.*

(a) The relation $\tau(n)\ll \log^A n$ is false for every constant A.
(b) The relation $\tau(n)\ll n^\delta$ is true for every fixed $\delta>0$.

PROOF. (a) Let n be any of the numbers $(2\cdot 3\cdots p_r)^m,\ m=1,2,\cdots$; here r is arbitrary but fixed. Then

$$\tau(n)=\prod_{p|n}(m+1)=(m+1)^r>m^r.$$

But $m=\log n/\log(2\cdot 3\cdots p_r)$, so that

$$\tau(n)>\frac{\log^r n}{(\log(2\cdot 3\cdots p_r))^r}\gg \log^r n,$$

where the implied constant depends only on r, and not on n.

(b) Let

$$f(n)=\frac{\tau(n)}{n^\delta};$$

the f is multiplicative. But $f(p^m)=(m+1)/p^{m\delta}$, so that $f(p^m)\to 0$ as $p^m\to\infty$, that is, as either p or m, or both, increases. This clearly implies that $f(n)\to 0$ as $n\to\infty$, which is the assertion. $\square$

REMARK 3.7. The above argument can be pushed a little further. We can give the inequality in (b) with an explicit constant for each δ.

REMARK 3.8. Here we can also consider the function $\tau_k(n)$ which denotes the number of (ordered) factorization of n as the product of exactly k positive integers.

THEOREM 3.9. *We have*

$$\varphi(n) \gg \frac{n}{\log\log n}.$$

PROOF. By Theorem 2.21, we have

$$\frac{\varphi(n)}{n} = \prod_{p|n}\left(1 - \frac{1}{p}\right)$$

so that

$$\log\frac{\varphi(n)}{n} = \sum_{p|n}\log\left(1-\frac{1}{p}\right) = -\sum_{p|n}\frac{1}{p} + \sum_{p|n}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right).$$

Since we have $\log\left(1-\frac{1}{p}\right)+\frac{1}{p} \ll \frac{1}{p^2}$, we have

$$\sum_{p|n}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right) \ll 1.$$

We write

$$\sum_{p|n}\frac{1}{p} = \sum_{\substack{p|n\\ p\le\log n}}\frac{1}{p} + \sum_{\substack{p|n\\ p\ge\log n}}\frac{1}{p}.$$

The total number of prime factors greater than $\log n$ of n with multiplicity is $O(\frac{\log n}{\log\log n})$. Hence we get

$$\sum_{\substack{p|n\\ p\ge\log n}}\frac{1}{p} \ll \frac{1}{\log\log n}.$$

By Mertens' Theorem 4.6, we have

$$\sum_{\substack{p|n\\ p\le\log n}}\frac{1}{p} \le \sum_{p\le\log n}\frac{1}{p} = \log\log\log n + O(1).$$

Hence we obtain

$$\sum_{p|n}\frac{1}{p} \le \log\log\log n + O(1).$$

Thus we have

$$\log\frac{\varphi(n)}{n} \ge -\log\log\log n + O(1)$$

and

$$\varphi(n) \gg \frac{n}{\log\log n}.$$

□

REMARK 3.10. If we do not use Mertens' Theorem 4.6, then we can prove

$$\varphi(n) \gg \frac{n}{\log n},$$

by applying

$$\sum_{\substack{p|n \\ p\le\log n}} \frac{1}{p} \le \sum_{p\le\log n} \frac{1}{p} \le \sum_{m\le\log n} \frac{1}{m} = \log\log n + O(1),$$

which follows from Theorem 3.4.

REMARK 3.11. The result of Theorem 3.9 is best-possible, in the sense that there is an increasing sequence of positive integers $n_1, n_2, \cdots$ such that

$$\varphi(n_r) \ll \frac{n_r}{\log\log n_r}.$$

PROOF. Take $n_r = p_1 p_2 \cdots p_r$, where p_r is the r-th prime. By Chebyshev inequality we have

$$\log n_r = \sum_{p\le p_r} \log p \asymp p_r.$$

Note that

$$\begin{aligned}\log \frac{\varphi(n_r)}{n_r} &= \log \prod_{p\le p_r}\left(1-\frac{1}{p}\right) = \sum_{p\le p_r} \log\left(1-\frac{1}{p}\right)\\ &= -\sum_{p\le p_r}\frac{1}{p} + \sum_{p\le p_r}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right) = -\log\log p_r + O(1).\end{aligned}$$

Hence

$$\varphi(n_r) \asymp \frac{n_r}{\log p_r} \asymp \frac{n_r}{\log\log n_r}.$$

□

THEOREM 3.12. *We have*

$$\frac{1}{2} < \frac{\sigma(n)\varphi(n)}{n^2} < 1.$$

PROOF. By Theorem 2.15 we have

$$\sigma(n) = \prod_{p^a\|n} \frac{p^{a+1}-1}{p-1} = n\prod_{p^a\|n}(1+p^{-1}+p^{-2}+\cdots+p^{-a}).$$

Hence

$$\sigma(n)\varphi(n) = n^2 \prod_{p^a\|n}(1-p^{-1})(1+p^{-1}+p^{-2}+\cdots+p^{-a}) = n^2\prod_{p^a\|n}(1-p^{-a-1}).$$

We have

$$\sigma(n)\varphi(n) < n^2$$

and

$$\sigma(n)\varphi(n) \ge n^2\prod_{p|n}(1-p^{-2}) > n^2\frac{1}{\zeta(2)} > \frac{n^2}{2}.$$

Here we have used $\zeta(2) = \frac{\pi^2}{6} < 2$. □

THEOREM 3.13. *We have*

$$\sigma(n) \ll n\log\log n.$$

PROOF. By Theorem 3.12 we have

$$\sigma(n) \ll \frac{n^2}{\varphi(n)}.$$

By Theorem 3.9 we obtain

$$\sigma(n) \ll \frac{n^2}{n/\log\log n} \ll n \log\log n. \qquad \square$$

3.4. Averages of $\tau(n)$, $\sigma(n)$, $\varphi(n)$ and $|\mu|(n)$

At first, we introduce Dirichlet's trick of switching divisors (the hyperbola method).

THEOREM 3.14 (the hyperbola method). *Let $h = f * g$, and*

$$F(x) = \sum_{n \le x} f(n), \quad G(x) = \sum_{n \le x} g(n), \quad H(x) = \sum_{n \le x} h(n),$$

then for any $y \in [1, x]$, we have

$$H(x) = \sum_{n \le y} f(n) G\left(\frac{x}{n}\right) + \sum_{m \le x/y} g(m) F\left(\frac{x}{m}\right) - F(y) G\left(\frac{x}{y}\right).$$

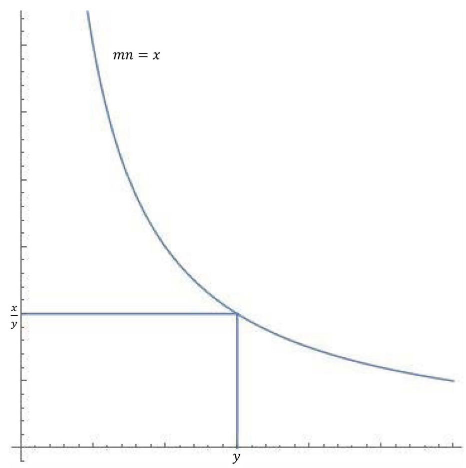

FIGURE 2.

PROOF. Since $h = f * g$, we have

$$\begin{aligned} H(x) &= \sum_{k} \sum_{mn \le x} f(n) g(m) \\ &= \sum_{n \le y} f(n) \sum_{m \le x/n} g(m) + \sum_{m \le x/y} g(m) \sum_{n \le x/m} f(n) - \sum_{n \le y} f(n) \sum_{m \le x/y} g(m) \\ &= \sum_{n \le y} f(n) G\left(\frac{x}{n}\right) + \sum_{m \le x/y} g(m) F\left(\frac{x}{m}\right) - F(y) G\left(\frac{x}{y}\right). \end{aligned}$$

This concludes the proof. $\square$

THEOREM 3.15. *We have*

$$\sum_{n\le x}\tau(n)=x\log x+(2\gamma-1)x+O(x^{1/2}),$$

where γ is Euler's constant.

PROOF. Note that $\tau=u*u$. By Theorem 3.14 with $y=\sqrt{x}$, we have

$$\sum_{n\le x}\tau(n)=2\sum_{n\le\sqrt{x}}\left[\frac{x}{n}\right]-\left[\sqrt{x}\right]^2.$$

By Theorem 3.4 we have

$$\begin{aligned}\sum_{n\le x}\tau(n)&=2x\sum_{n\le\sqrt{x}}\frac{1}{n}-x+O(x^{1/2})\\&=2x(\log x+\gamma+O(x^{-1/2}))-x+O(x^{1/2})\\&=x\log x+(2\gamma-1)x+O(x^{1/2}),\end{aligned}$$

as claimed. □

THEOREM 3.16. *We have*

$$\sum_{n\le x}\varphi(n)=\frac{3x^2}{\pi^2}+O(x\log x).$$

PROOF. Note that $\varphi=\mu*N$ and

$$\sum_{n\le y}n=\frac{y^2}{2}+O(y).$$

We have

$$\begin{aligned}\sum_{n\le x}\varphi(n)&=\sum_{n\le x}\mu(n)\sum_{m\le\frac{x}{n}}m=\sum_{n\le x}\mu(n)\left(\frac{x^2}{2n^2}+O\left(\frac{x}{n}\right)\right)\\&=\frac{x^2}{2}\sum_{n\le x}\frac{\mu(n)}{n^2}+O(x\log x)=\frac{x^2}{2}\sum_{n\ge1}\frac{\mu(n)}{n^2}+O(x\log x).\end{aligned}$$

Note that $\sum_{n\ge1}\frac{\mu(n)}{n^2}=\frac{1}{\zeta(2)}=\frac{6}{\pi^2}$. This completes the proof. □

THEOREM 3.17. *We have*

$$\sum_{n\le x}\sigma(n)=\frac{\pi^2x^2}{12}+O(x\log x).$$

PROOF. Note that $\sigma=u*N$. We have

$$\begin{aligned}\sum_{n\le x}\sigma(n)&=\sum_{n\le x}\sum_{m\le\frac{x}{n}}m=\sum_{n\le x}\left(\frac{x^2}{2n^2}+O\left(\frac{x}{n}\right)\right)\\&=\frac{x^2}{2}\sum_{n\le x}\frac{1}{n^2}+O(x\log x)=\frac{x^2}{2}\sum_{n\ge1}\frac{1}{n^2}+O(x\log x).\end{aligned}$$

Note that $\sum_{n\ge1}\frac{1}{n^2}=\zeta(2)=\frac{\pi^2}{6}$. This completes the proof. □

THEOREM 3.18. *Let $Q(x)$ be the number of positive square-free integers not exceeding x. We have*

$$Q(x) = \sum_{n \le x} |\mu|(n) = \frac{6}{\pi^2} x + O(\sqrt{x}).$$

PROOF. By Theorem 2.19, we have $|\mu|(n) = \sum_{d^2 | n} \mu(d)$. Hence we have

$$\begin{aligned}\sum_{n \le x} |\mu|(n) &= \sum_{d \le \sqrt{x}} \mu(d) \sum_{m \le \frac{x}{d^2}} 1 = \sum_{d \le \sqrt{x}} \mu(d) \left(\frac{x}{d^2} + O(1) \right) \\ &= x \sum_{n \le \sqrt{x}} \frac{\mu(n)}{n^2} + O(\sqrt{x}) = x \sum_{n \ge 1} \frac{\mu(n)}{n^2} + O(\sqrt{x}) = \frac{6}{\pi^2} x + O(\sqrt{x}),\end{aligned}$$

where we have used $\sum_{n \ge 1} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2}$. □

3.5. The Gauss circle problem and the Dirichlet divisor problem

DEFINITION 3.19. The sum of two squares function $r_2(n)$ is defined as the number of pairs of integers (a, b) such that $n = a^2 + b^2$, that is,

$$r_2(n) = \#\{a, b \in \mathbb{Z} : a^2 + b^2 = n\}.$$

The Gauss circle problem is the problem of determining how many integer lattice points there are in a circle centered at the origin and with radius $\sqrt{x}$. This number is approximated by the area of the circle, so the real problem is to accurately bound the error term describing how the number of points differs from the area. The first progress on a solution was made by Carl Friedrich Gauss, hence its name.

THEOREM 3.20 (Gauss). *We have*

$$\sum_{n \le x} r_2(n) = \pi x + O(x^{1/2}).$$

PROOF. We use the slicing method. By the definition of $r_2(n)$, we have

$$\begin{aligned}\sum_{n \le x} r_2(n) &= \sum_{|a| \le \sqrt{x}} \sum_{|b| \le \sqrt{x - a^2}} 1 = \sum_{|a| \le \sqrt{x}} (2[\sqrt{x - a^2}] + O(1)) \\ &= 4 \sum_{1 \le a \le \sqrt{x}} \sqrt{x - a^2} + O(\sqrt{x}).\end{aligned}$$

By Theorem 3.5 with $\lambda_n = n$, $c_n = 1$ and $f(n) = \sqrt{x - n^2}$, we get

$$\begin{aligned}\sum_{n \le x} r_2(n) &= -4 \int_1^{\sqrt{x}} \frac{-u}{\sqrt{x - u^2}} (u + O(1)) \mathrm{d}u + O(\sqrt{x}) \\ &= 4\sqrt{x} \int_0^{\frac{\pi}{2}} \sin t \, (\sqrt{x} \sin t + O(1)) \mathrm{d}t + O(\sqrt{x}) \\ &= 4x \int_0^{\frac{\pi}{2}} (\sin t)^2 \mathrm{d}t + O(\sqrt{x}).\end{aligned}$$

Here we made a change of variable $u = \sqrt{x}\sin t$. Note that

$$\int_0^{\frac{\pi}{2}} (\sin t)^2 \mathrm{d}t = \int_0^{\frac{\pi}{2}} \frac{1-\cos 2t}{2}\mathrm{d}t$$
$$= \frac{\pi}{4} - \frac{1}{4}\int_0^{\pi} \cos\theta \mathrm{d}\theta = \frac{\pi}{4}.$$

Hence we get

$$\sum_{n\le x} r_2(n) = \pi x + O(x^{1/2}),$$

as claimed. □

By harmonic analysis we can improve the exponent to 1/3. Furthermore, applying the exponential sum method we can even go lower than 1/3. The best exponent in the error term is conjectured to be 1/4 and it is well known that one can not do better than 1/4.

THEOREM 3.21 (Voronoi, Sierpinski). *We have*

$$\sum_{n\le x} r_2(n) = \pi x + O(x^{1/3}).$$

To prove Theorem 3.21, we will use the Poisson summation formulas. Let $L^1(\mathbb{R})$ denote the space of Lebesgue integrable functions on $\mathbb{R}$. Recall that for any function $f \in L^1(\mathbb{R})$ its Fourier transform is defined by

$$\hat{f}(y) = \int_{\mathbb{R}} f(x)e(-xy)\mathrm{d}x.$$

We first give the follow Poisson summation formula on $\mathbb{R}$.

THEOREM 3.22. *Suppose that both f and $\hat{f}$ are in $L^1(\mathbb{R})$ and have bounded variation. Then we have*

$$\sum_{m\in\mathbb{Z}} f(m) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

where both series converge absolutely.

PROOF. Consider the function

$$F(x) = \sum_{m\in\mathbb{Z}} f(m+x)$$

which is periodic of period one. This has the absolutely convergent Fourier series expansion

$$F(x) = \sum_{n\in\mathbb{Z}} c_F(n)e(nx)$$

with coefficients given by

$$c_F(n) = \int_0^1 F(u)e(-nu)\mathrm{d}u = \int_{-\infty}^{\infty} f(u)e(-nu)\mathrm{d}u = \hat{f}(n).$$

Taking $F(0)$ we get the Poisson summation formula. □

The same method (averaging of integral translations) works in several variables giving

THEOREM 3.23. *Suppose f in the Schwartz class $\mathcal{S}(\mathbb{R}^\ell)$. Then we have*

$$\sum_{m\in\mathbb{Z}^\ell} f(m) = \sum_{n\in\mathbb{Z}^\ell} \hat{f}(n).$$

LEMMA 3.24. *Let $f : \mathbb{R}^2 \to \mathbb{C}$. Suppose*

$$f(x) = g(|x|^2)$$

where g is a smooth compactly supported function on $\mathbb{R}^+$. Then we have

$$\hat{f}(y) = h(|y|^2)$$

with

$$h(y) = \pi \int_0^\infty J_0(2\pi\sqrt{xy})g(x)\mathrm{d}x$$

where $J_0(x)$ is the Bessel function of order 0.

PROOF. For integer values of ν, the Bessel function has the following integral representation:

$$J_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(\nu\theta - z\sin\theta)\mathrm{d}\theta.$$

Hence $J_0(z) = \frac{1}{\pi}\int_0^\pi \cos(-z\sin\theta)\mathrm{d}\theta$. Since $f(x) = g(|x|^2)$, by using the polar coordinates we have

$$\hat{f}(y) = \int_{\mathbb{R}^2} f(x)e(-xy)\mathrm{d}x = \int_0^\infty g(r^2)\int_0^{2\pi} e(-r\cos(\theta)y_1 - r\sin(\theta)y_2)r\mathrm{d}r\mathrm{d}\theta.$$

If $y = 0$ then $\hat{f}(0) = \int_{\mathbb{R}^2} f(x)\mathrm{d}x = 2\pi\int_0^\infty g(r^2)r\mathrm{d}r = \pi\int_0^\infty g(x)\mathrm{d}x$ and $h(0) = \pi\int_0^\infty g(x)\mathrm{d}x = \hat{f}(0)$. Here we have used $J_0(0) = 1$. Now we assume $y \neq 0$. Let $\varphi \in [0, 2\pi]$ such that $\cos\varphi = \frac{y_1}{|y|}$ and $\sin(\varphi) = \frac{y_2}{|y|}$. Then we have

$$\begin{aligned}\hat{f}(y) &= \int_0^\infty g(r^2)\int_0^{2\pi} e(-r\cos(\theta+\varphi)y_1 - r\sin(\theta+\varphi)y_2)r\mathrm{d}r\mathrm{d}\theta \\ &= \int_0^\infty g(r^2)\int_0^{2\pi} e(-r\cos(\theta)|y|)r\mathrm{d}r\mathrm{d}\theta = \int_0^\infty g(r^2)\int_0^{2\pi} e(-r\sin(\theta)|y|)r\mathrm{d}r\mathrm{d}\theta.\end{aligned}$$

Making a change of variable $r^2 = x$ we have

$$\hat{f}(y) = \int_0^\infty g(x)\int_0^\pi e(-\sqrt{x}\sin(\theta)|y|)\mathrm{d}x\mathrm{d}\theta = h(|y|^2),$$

as claimed. □

LEMMA 3.25. *We have*

$$h(y) = -\frac{1}{\pi y}\int_0^\infty (uy)^{1/4} g'(u)\sin\left(2\pi\sqrt{uy} - \frac{\pi}{4}\right)\mathrm{d}u + O(R(y)),$$

where

$$R(y) = \int_0^\infty (uy)^{-5/4}(|g(u)| + u|g'(u)|)\mathrm{d}u.$$

PROOF. We have the following asymptotic expansion:

$$\pi J_0(z) = \left(\frac{2\pi}{z}\right)^{1/2} \left\{\cos\left(z - \frac{\pi}{4}\right) + \frac{1}{8z}\sin\left(z - \frac{\pi}{4}\right) + O\left(\frac{1}{z^2}\right)\right\} \tag{3.1}$$

valid for $z > 0$. We get

$$\begin{aligned} h(y) = &\int_0^\infty g(x)(xy)^{-1/4}\cos(2\pi\sqrt{xy} - \pi/4)\mathrm{d}x \\ &+ \int_0^\infty g(x)\frac{(xy)^{-3/4}}{16\pi}\sin(2\pi\sqrt{xy} - \pi/4)\mathrm{d}x + O\left(\int_0^\infty |g(x)|(xy)^{-5/4}\mathrm{d}x\right). \end{aligned}$$

Integrating by parts in the second integral above we obtain

$$\begin{aligned} \int_0^\infty g(x)\frac{(xy)^{-3/4}}{16\pi}\sin(2\pi\sqrt{xy} - \pi/4)\mathrm{d}x &= \int_0^\infty g(x)\frac{(xy)^{-3/4}}{16\pi}\frac{-x}{\pi\sqrt{xy}}\mathrm{d}\cos(2\pi\sqrt{xy} - \pi/4) \\ &\ll R(y) \end{aligned}$$

and hence

$$h(y) = \int_0^\infty g(x)(xy)^{-1/4}\cos(2\pi\sqrt{xy} - \pi/4)\mathrm{d}x + O(R(y)).$$

Now integrating by parts in the integral above we get

$$h(y) = -\frac{1}{\pi y}\int_0^\infty (uy)^{1/4}g'(u)\sin\left(2\pi\sqrt{uy} - \frac{\pi}{4}\right)\mathrm{d}u + O(R(y)).$$

This completes the proof. □

COROLLARY 3.26. *Suppose g is a smooth and compactly supported on $\mathbb{R}^+$. Then*

$$\sum_{n=0}^\infty r_2(m)g(m) = \pi\int_0^\infty g(x)\mathrm{d}x + \sum_{n=1}^\infty r_2(n)h(n)$$

where

$$h(y) = \pi\int_0^\infty g(x)J_0(2\pi\sqrt{xy})\mathrm{d}x$$

and both series converge absolutely.

PROOF. By Theorem 3.23 with $f(x) = g(|x|^2)$, we have

$$\sum_{n=0}^\infty r_2(n)g(n) = h(0) + \sum_{n=1}^\infty r_2(n)h(n).$$

Noting that $h(0) = \pi\int_0^\infty g(x)\mathrm{d}x$, we completes the proof. □

PROOF OF THEOREM 3.21. Let $0 < y \le x/5$. Let w_1 be a smooth function with support $\operatorname{supp} w_1 \in [0, 1 + y/x]$ such that $w_1(u) = 1$ if $u \in [y/x, 1]$, $w_1(u) \in [0,1]$ if $u \in [0, y/x] \cup [1, 1 + y/x]$. Similarly, let w_2 be a smooth function with support $\operatorname{supp} w_2 \in [0, 1]$ such that $w_2(u) = 1$ if $u \in [y/x, 1 - y/x]$, $w_2(u) \in [0,1]$ if $u \in [0, y/x] \cup [1 - y/x, 1]$. Assume $w_j^{(k)}(u) \ll (x/y)^k$ for any integer $k \ge 0$ and $j \in \{1, 2\}$. By the fact $r_2(n) \ge 0$ and Theorem 3.20, we have

$$\sum_{n\ge1} r_2(n)w_2\left(\frac{n}{x}\right) \le \sum_{n\le x} r_2(n) \le \sum_{n\ge1} r_2(n)w_1\left(\frac{n}{x}\right) + y.$$

In order to prove Theorem 3.21, it suffices to prove for $w \in \{w_1, w_2\}$ and $y = x^{1/3}$, we have

$$\sum_{n\geq 1} r_2(n) w\left(\frac{n}{X}\right) = \pi\tilde{w}(1)x + O(x^{1/3}). \tag{3.2}$$

where $\tilde{w}(s) = \int_0^\infty w(u)u^{s-1}\mathrm{d}u$ is the Mellin transform of w. Note that $\tilde{w}(1) = \int_0^\infty w(u)\mathrm{d}u = 1 + O(y/x)$. Indeed, (3.2) leads to

$$\sum_{n\geq 1} r_2(n) w\left(\frac{n}{X}\right) = \pi x + O(x^{1/3}),$$

as claimed.

Now we prove (3.2). By Corollary 3.26 we have

$$\sum_{n\geq 1} r_2(n) w\left(\frac{n}{x}\right) = \pi\int_0^\infty w(u/x)\mathrm{d}u + \sum_{n=1}^\infty r_2(n)h(n).$$

with $g(u) = w(u/x)$. Note that $g'(u) = w'(u/x)\frac{1}{x}$. By Lemma 3.25 we have

$$h(n) = -\frac{1}{\pi n}\int_0^\infty (un)^{1/4} w'(u/x)\frac{1}{x}\sin\left(2\pi\sqrt{un} - \frac{\pi}{4}\right)\mathrm{d}u + O(R(n)),$$

where

$$R(n) = \int_0^\infty (un)^{-5/4}(|w(u/x)| + u|w'(u/x)\frac{1}{x}|)\mathrm{d}u.$$

Note that $w(u/x) \ll 1$, $w(u/x) = w(0) + \int_0^{u/x} w'(v)\mathrm{d}v \ll u/y$ for $u \in [0, y]$, and $w'(u/x) \ll x/y$ for $u \in [0, y] \cup [x - y, x + y]$ and $w'(u/x) = 0$ for $u \in [y, 1 - y]$. Hence

$$\begin{aligned}
R(n) \ll{}& \int_0^y (un)^{-5/4}\frac{u}{y}\mathrm{d}u + \int_y^{x+y}(un)^{-5/4}\mathrm{d}u \\
&+ \int_0^y (un)^{-5/4}\frac{u}{x}|w'(u/x)|\mathrm{d}u + \int_{x-y}^{x+y}(un)^{-5/4}\frac{u}{x}|w'(u/x)|\mathrm{d}u \ll n^{-5/4}y^{-1/4},
\end{aligned}$$

$$\begin{aligned}
h(n) &= -\frac{1}{\pi n}\int_0^\infty (un)^{1/4}w'(u/x)\frac{1}{x}\sin\left(2\pi\sqrt{un} - \frac{\pi}{4}\right)\mathrm{d}u + O(R(n)) \\
&\ll n^{-3/4}\int_0^y u^{1/4}\frac{1}{x}|w'(u/x)|\mathrm{d}u + n^{-3/4}\int_{x-y}^{x+y} u^{1/4}\frac{1}{x}|w'(u/x)|\mathrm{d}u + O(R(n)) \\
&\ll n^{-3/4}x^{1/4} + n^{-5/4}y^{-1/4},
\end{aligned}$$

and by integrating by parts,

$$\begin{aligned}
h(n) &= -\frac{x}{\pi n}\int_0^\infty (xn\xi)^{1/4}w'(\xi)\frac{1}{x}\sin\left(2\pi\sqrt{nx\xi} - \frac{\pi}{4}\right)\mathrm{d}\xi + O(R(n)) \\
&= \frac{x^{1/4}}{\pi n^{3/4}}\int_0^\infty \xi^{1/4}w'(\xi)\frac{\xi}{\pi\sqrt{xn\xi}}\mathrm{d}\cos\left(2\pi\sqrt{nx\xi} - \frac{\pi}{4}\right) + O(R(n)) \\
&\ll n^{-5/4}x^{-1/4}\left(\int_0^{y/x} + \int_{1-y/x}^{1+y/x}\right)\left(\xi^{3/4}|w''(\xi)| + \xi^{-1/4}|w'(\xi)|\right)\mathrm{d}\xi + O(R(n)) \\
&\ll n^{-5/4}y^{-1/4} + n^{-5/4}x^{3/4}/y.
\end{aligned}$$

Hence we obtain

$$h(n) \ll \min\{n^{-3/4}x^{1/4}, n^{-5/4}x^{3/4}/y\}.$$

Thus we get

$$\sum_{n\geq 1} r_2(n)w\left(\frac{n}{x}\right) = \pi\int_0^\infty w(u/x)\mathrm{d}u + \sum_{n\leq x/y^2} r_2(n)n^{-3/4}x^{1/4} + \sum_{n\geq x/y^2} r_2(n)n^{-5/4}x^{3/4}/y.$$

By the partial summation formula we get

$$\sum_{n\geq 1} r_2(n)w\left(\frac{n}{x}\right) = \pi\int_0^\infty w(u/x)\mathrm{d}u + O(x^{1/2}/y^{1/2}).$$

Therefore, we have

$$\begin{aligned}\sum_{n\leq x} r_2(n) &= \sum_{n\geq 1} r_2(n)w\left(\frac{n}{x}\right) + O(y)\\ &= \pi x + O(x^{1/2}/y^{1/2} + y) = \pi x + O(x^{1/3}),\end{aligned}$$

by taking $y = x^{1/3}$. This proves the theorem. □

The Dirichlet divisor problem is to accurately bound the error term in the average of $\tau(n)$. This is closed related to the Gauss circle problem. We state the following theorem without giving a proof.

THEOREM 3.27. *We have*

$$\sum_{n\leq x} \tau(n) = x\log x + (2\gamma - 1)x + O(x^{1/3}\log x),$$

where γ *is Euler's constant.*