Homework 3

(due Wednesday, April 14, 2021)

1. Show that
$$\prod_{2<p\le x}\left(1-\frac{2}{p}\right)=\frac{A+o(1)}{\log^2 x},$$
where A is an absolute constant.

2. Let $\alpha\in\mathbb{R}$ be a parameter. Discuss when the series
$$\sum_p \frac{1}{p(\log\log p)^\alpha}$$
converges and when it diverges.

3. Let f be an arithmetic function such that
$$\sum_{p\le x} f(p)\log p=(ax+b)\log x+cx+O(1),\quad \text{for } x\ge 2.$$
Prove that there is a constant A depending on f such that for $x\ge 2$,
$$\sum_{p\le x} f(p)=ax+(a+c)\left(\frac{x}{\log x}+\int_2^x\frac{\mathrm{d}t}{\log^2 t}\right)+b\log\log x+A+O\left(\frac{1}{\log x}\right).$$

4. If $\psi(x)/x$ tends to a limit as $x\to\infty$, then this limit equals 1. Here $\psi(x)=\sum_{n\le x}\Lambda(n)$.

5. Show that $\pi(x)\sim\frac{x}{\log x}$ is equivalent to $\sum_{n\ge 1}\frac{\mu(n)}{n}=0$. Here $\pi(x)=\sum_{p\le x}1$.

6. Prove that the following two statements are equivalent:
$$\pi(x)=\frac{x}{\log x}+O\left(\frac{x}{\log^2 x}\right); \tag{1}$$
$$\theta(x)=x+O\left(\frac{x}{\log x}\right). \tag{2}$$
Here $\theta(x)=\sum_{p\le x}\log p$.

7. Let
$$\mathrm{Li}(x)=\int_2^x\frac{\mathrm{d}t}{\log t}.$$
Show that for each fixed positive integer n, we have
$$\mathrm{Li}(x)=\frac{x}{\log x}+\frac{1!x}{\log^2 x}+\frac{2!x}{\log^3 x}+\cdots+\frac{(n-1)!x}{\log^n x}+O\left(\frac{x}{\log^{n+1} x}\right).$$