CHAPTER 4

The Distribution of Primes

To study prime numbers, Chebyshev introduced the following two functions

$$\psi(x) = \sum_{n \le x} \Lambda(n), \quad \theta(x) = \sum_{p \le x} \log p.$$

THEOREM 4.1. *The following statements are equivalent:*

$$\pi(x) \sim \frac{x}{\log x}, \quad x \to \infty, \tag{4.1}$$

$$\psi(x) \sim x, \quad x \to \infty, \tag{4.2}$$

$$\theta(x) \sim x, \quad x \to \infty. \tag{4.3}$$

PROOF. We first show that (4.2) $\Longleftrightarrow$ (4.3). By definition of Λ, we have

$$\psi(x) = \sum_{k \le \log_2 x} \sum_{p^k \le x} \log p = \sum_{k \le \log_2 x} \theta\left(x^{1/k}\right).$$

Note that $\theta(y) \le y \log y$. So we have

$$\psi(x) - \theta(x) = \sum_{2 \le k \le \log_2 x} \theta\left(x^{1/k}\right) \ll x^{1/2} \log x,$$

from which we get (4.2) $\Longleftrightarrow$ (4.3).

Now we show (4.1) $\Longrightarrow$ (4.3). By Theorem 3.5 with $\lambda_n = n$, $f(u) = \log u$, $c_n = \delta_{n \in \mathbb{P}} = \begin{cases} 1, & \text{if } n \in \mathbb{P}, \\ 0, & \text{otherwise} \end{cases}$, we have

$$\theta(x) = \pi(x) \log x - \int_2^x \frac{1}{u} \pi(u) \mathrm{d}u.$$

If $\pi(x) \sim x / \log x$, then we have

$$\int_2^x \frac{1}{u}\pi(u)\mathrm{d}u = \int_2^{\sqrt{x}} \frac{1}{u}\pi(u)\mathrm{d}u + \int_{\sqrt{x}}^x \frac{1}{u}\pi(u)\mathrm{d}u \ll \sqrt{x} + \int_{\sqrt{x}}^x \frac{1}{\log u}\mathrm{d}u \ll \frac{x}{\log x},$$

and $\pi(x) \log x \sim x$. Hence we get (4.3).

Finally we show (4.1) $\Longleftarrow$ (4.3). By Theorem 3.5 with $\lambda_n = n$, $f(u) = 1/\log u$, $c_n = \delta_{n \in \mathbb{P}}$, we have

$$\pi(x) = \frac{\theta(x)}{\log x} - \int_2^x \frac{1}{u(\log u)^2} \theta(u) \mathrm{d}u.$$

If $\theta(x) \sim x$, then we have

$$\int_2^x \frac{1}{u(\log u)^2}\theta(u)\mathrm{d}u = \int_2^{\sqrt{x}} \frac{\theta(u)}{u(\log u)^2}\mathrm{d}u + \int_{\sqrt{x}}^x \frac{\theta(u)}{u(\log u)^2}\mathrm{d}u\mathrm{d}u$$
$$\ll \sqrt{x} + \int_{\sqrt{x}}^x \frac{1}{(\log u)^2}\mathrm{d}u \ll \frac{x}{(\log x)^2},$$

and $\frac{\theta(x)}{\log x} \sim \frac{x}{\log x}$. Hence we get (4.1). This completes the proof. □

4.1. The Chebyshev inequality and Mertens' theorems

The prime number theorem states that $\pi(x) \sim x/\log x$ as $x \to \infty$. In this section, we show that $x/\log x$ is the correct order of magnitude of $\pi(x)$. We first consider Chebyshev's ψ-function.

THEOREM 4.2. *We have*

$$\psi(x) \asymp x.$$

PROOF. By Theorem 2.46, we have

$$S(x) := \sum_{n\le x} \log n = \sum_{ab\le x} \Lambda(b) = \sum_{a\le x} \psi\left(\frac{x}{a}\right).$$

Hence we have

$$S(x) - 2S(x/2) = \sum_{a\le x} \psi\left(\frac{x}{a}\right) - 2\sum_{a\le x/2} \psi\left(\frac{x}{2a}\right) = \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{3}\right) - \cdots \le \psi(x),$$

and

$$S(x) - 2S(x/2) \ge \psi(x) - \psi\left(\frac{x}{2}\right).$$

By Theorem 3.3, we have

$$S(x) = \sum_{n\le x} \log n = x\log x - x + O(\log x).$$

Hence

$$S(x) - 2S(x/2) = x\log x - x - x\log\frac{x}{2} + x + O(\log x) = x\log 2 + O(\log x) \le \psi(x),$$

and

$$\psi(x) - \psi\left(\frac{x}{2}\right) \le x\log 2 + O(\log x).$$

So

$$\psi\left(\frac{x}{2}\right) - \psi\left(\frac{x}{4}\right) \le \frac{x}{2}\log 2 + O(\log x),$$
$$\psi\left(\frac{x}{4}\right) - \psi\left(\frac{x}{8}\right) \le \frac{x}{4}\log 2 + O(\log x), \ldots.$$

Therefore, we have

$$\psi(x) \le \sum_{k\ge 0} \frac{1}{2^k}\, x\log 2 + O((\log x)^2) \le (2\log 2)x + O((\log x)^2).$$

Hence we have

$$x\log 2 + O(\log x) \le \psi(x) \le (2\log 2)x + O((\log x)^2).$$

This completes the proof. □

THEOREM 4.3. *We have*

$$\pi(x) \asymp \frac{x}{\log x}.$$

PROOF. By Theorem 4.2, we have

$$\pi(x) \geq \frac{1}{\log x}\sum_{p\leq x}\log p = \frac{1}{\log x}\left(\psi(x) - \sum_{k\geq 2}\sum_{p^k\leq x}\log p\right) \gg \frac{x}{\log x},$$

By Theorems 3.5 and 4.2, we have

$$\pi(x) = \sum_{p\leq x}\frac{\log p}{\log p} \leq \int_2^x \frac{1}{\log u}\mathrm{d}\sum_{p\leq u}\log p \leq \frac{\psi(u)}{\log u}\Big|_2^x + \int_2^x \psi(u)\frac{1}{u(\log u)^2}\mathrm{d}u$$

$$\ll \frac{x}{\log x} + \int_2^x \frac{1}{(\log u)^2}\mathrm{d}u \ll \frac{x}{\log x} + \int_2^{\sqrt{x}} \mathrm{d}u + \int_{\sqrt{x}}^x \frac{1}{(\log x)^2}\mathrm{d}u \ll \frac{x}{\log x}.$$

This completes the proof. □

COROLLARY 4.4. *For $n \geq 1$, the n-th prime p_n satisfies the inequalities*

$$p_n \asymp n\log n.$$

PROOF. By Theorem 4.3, we have

$$n = \pi(p_n) \asymp \frac{p_n}{\log p_n}.$$

Note that $p_n \geq n$. We have $p_n \gg n\log p_n \gg n\log n$.

Note that $p_n \ll n\log p_n \ll np_n^{1/2}$. Hence $p_n \ll n^2$. Therefore, we have $p_n \ll n\log n^2 \ll n\log n$. □

THEOREM 4.5. *We have*

$$\sum_{n\leq x}\frac{\Lambda(n)}{n} = \log x + O(1),$$

and

$$\sum_{p\leq x}\frac{\log p}{p} = \log x + O(1).$$

PROOF. By Theorems 2.46 and 4.2, we have

$$S(x) = \sum_{n\leq x}\log n = \sum_{ab\leq x}\Lambda(b) = \sum_{b\leq x}\Lambda(b)\left(\frac{x}{b} + O(1)\right) = x\sum_{n\leq x}\frac{\Lambda(n)}{n} + O(x).$$

By Theorem 3.3, we have

$$S(x) = x\log x + O(x) = x\sum_{n\leq x}\frac{\Lambda(n)}{n} + O(x).$$

Hence we get

$$\sum_{n\leq x}\frac{\Lambda(n)}{n} = \log x + O(1)$$

from which we obtain

$$\sum_{p\le x}\frac{\log p}{p}=\sum_{n\le x}\frac{\Lambda(n)}{n}-\sum_{k\ge 2}\sum_{p^k\le x}\frac{\log p}{p^k}=\log x+O(1). \tag{4.4}$$

This completes the proof. □

THEOREM 4.6. *We have*

$$\sum_{p\le x}\frac{1}{p}=\log\log x+C+O\left(\frac{1}{\log x}\right),$$

for some constant C.

PROOF. By Theorem 3.5, we have

$$\sum_{p\le x}\frac{1}{p}=\int_2^x\frac{1}{\log u}\mathrm{d}\sum_{p\le u}\frac{\log p}{p}=\frac{1}{\log x}\sum_{p\le x}\frac{\log p}{p}-\int_2^x\sum_{p\le u}\frac{\log p}{p}\mathrm{d}\frac{1}{\log u}.$$

By (4.4), we have

$$\frac{1}{\log x}\sum_{p\le x}\frac{\log p}{p}=1+O\left(\frac{1}{\log x}\right),$$

and

$$\begin{aligned}-\int_2^x\sum_{p\le u}\frac{\log p}{p}\mathrm{d}\frac{1}{\log u}&=-\int_2^x\log u\,\mathrm{d}\frac{1}{\log u}+\int_2^x\left(\log u-\sum_{p\le u}\frac{\log p}{p}\right)\,\mathrm{d}\frac{1}{\log u}\\&=-\int_2^x\log u\,\mathrm{d}\frac{1}{\log u}+C'-\int_x^\infty\left(\log u-\sum_{p\le u}\frac{\log p}{p}\right)\,\mathrm{d}\frac{1}{\log u}\\&=\int_2^x\frac{1}{u\log u}\mathrm{d}u+C'+O\left(\frac{1}{\log x}\right)\\&=\log\log u\Big|_2^x+C'+O\left(\frac{1}{\log x}\right),\end{aligned}$$

where $C'=\int_2^\infty\left(\log u-\sum_{p\le u}\frac{\log p}{p}\right)\,\mathrm{d}\frac{1}{\log u}$ is a constant. Hence

$$\sum_{p\le x}\frac{1}{p}=\log\log x+C+O\left(\frac{1}{\log x}\right),$$

where $C=C'+1-\log\log 2$. □

THEOREM 4.7. *Let $x\ge 2$. There exists a constant C such that*

$$\prod_{p\le x}\left(1-\frac{1}{p}\right)=\frac{A}{\log x}+O\left(\frac{1}{(\log x)^2}\right).$$

PROOF. Note that $\log(1+x)=x+O(x^2)$, as $x\to 0$. By Theorem 4.6 we have

$$\begin{aligned}\sum_{p\le x}\log\left(1-\frac{1}{p}\right)&=-\sum_{p\le x}\frac{1}{p}+\sum_{p\le x}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)\\&=-\log\log x-C+O\left(\frac{1}{\log x}\right)+B,\end{aligned}$$

where $B = \sum_{p\geq 2}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right) \ll \sum_{p\geq 2}\frac{1}{p^2} < \sum_{n\geq 2}\frac{1}{n^2} < \infty$. Therefore we obtain

$$\begin{aligned}\prod_{p\leq x}\left(1-\frac{1}{p}\right) &= \exp\left(\sum_{p\leq x}\log\left(1-\frac{1}{p}\right)\right)\\ &= \exp\left(-\log\log x + B - C + O\left(\frac{1}{\log x}\right)\right)\\ &= \frac{e^{B-C}}{\log x}\left(1+O\left(\frac{1}{\log x}\right)\right) = \frac{A}{\log x} + O\left(\frac{1}{(\log x)^2}\right),\end{aligned}$$

where $A = e^{B-C}$. □

4.2. Averages of $\omega(n)$ and $\Omega(n)$

THEOREM 4.8. *We have*

$$\sum_{n\leq x}\omega(n) = x\log\log x + cx + O\left(\frac{x}{\log x}\right)$$

and

$$\sum_{n\leq x}\Omega(n) = x\log\log x + c'x + o(x).$$

PROOF. By Theorems 4.3 and 4.6 we have

$$\begin{aligned}\sum_{n\leq x}\omega(n) &= \sum_{n\leq x}\sum_{p|n}1 = \sum_{p\leq x}\sum_{\substack{n\leq x\\ p|n}}1\\ &= \sum_{p\leq x}\left(\frac{x}{p}+O(1)\right)\\ &= x\log\log x + cx + O\left(\frac{x}{\log x}\right).\end{aligned}$$

Note that $\Omega(n) = \sum_{p^r\|n} r$. We have

$$\begin{aligned}\sum_{n\leq x}\Omega(n) &= \sum_{n\leq x}\sum_{p^r\|n} r = \sum_{p\leq x}\sum_{\substack{n\leq x\\ p^r\|n}} r\\ &= \sum_{p\leq x}\sum_{r=1}^{\infty}\sum_{\substack{n\leq x\\ p^r|n}}1 = \sum_{p\leq x}\sum_{\substack{n\leq x\\ p|n}}1 + \sum_{p\leq x^{1/2}}\sum_{\substack{r\geq 2\\ p^r\leq x}}\left(\frac{x}{p^r}+O(1)\right)\\ &= x\log\log x + cx + O\left(\frac{x}{\log x}\right) + \sum_{r\geq 2}\sum_{p^r\leq x}\frac{x}{p^r}\\ &= x\log\log x + \left(c + \sum_{r\geq 2}\sum_{p\geq 2}\frac{1}{p^r}\right)x + O\left(\frac{x}{\log x} + x\sum_{r\geq 2}\sum_{p^r>x}\frac{1}{p^r}\right).\end{aligned}$$

Note that

$$\sum_{r\geq 2}\sum_{p^r>x}\frac{1}{p^r} \ll \sum_{r\geq 2}\sum_{n^r>x}\frac{1}{n^r} \ll \sum_{r\leq \log_2 x}\int_{x^{1/r}}^{\infty} r\frac{1}{u^r}\mathrm{d}u + \sum_{r\geq \log_2 x}\int_{2}^{\infty} r\frac{1}{u^r}\mathrm{d}u$$
$$\ll \sum_{2\leq r\leq \log_2 x} x^{(-r+1)/r} + \sum_{r\geq \log_2 x} 2^{-r} \ll x^{-1/2}.$$

Hence we have

$$\sum_{n\leq x}\Omega(n) = x\log\log x + c'x + O\left(\frac{x}{\log x}\right),$$

with $c' = c + \sum_{r\geq 2}\sum_{p\geq 2}\frac{1}{p^r}$. □

THEOREM 4.9. *We have*

$$\sum_{n\leq x}\omega(n)^2 = x(\log\log x)^2 + O(x\log\log x).$$

PROOF. Let us consider the number of pairs of different prime factors p, q of n (i.e. $p \neq q$), counting the pair q, p distinct from p, q. There are $\omega(n)$ possible values of p and, with each of these, just $\omega(n) - 1$ possible values of q. Hence we have

$$\omega(n)(\omega(n)-1) = \sum_{\substack{pq|n\\ p\neq q}} 1 = \sum_{pq|n} 1 - \sum_{p^2|n} 1.$$

Summing over all $n \leq x$ we have

$$\sum_{n\leq x}\omega(n)(\omega(n)-1) = \sum_{n\leq x}\sum_{pq|n} 1 - \sum_{n\leq x}\sum_{p^2|n} 1$$
$$= \sum_{pq\leq x}\left[\frac{x}{pq}\right] - \sum_{p^2\leq x}\left[\frac{x}{p^2}\right] = \sum_{pq\leq x}\left[\frac{x}{pq}\right] + O(x).$$

By Theorem 4.6 we have

$$\sum_{pq\leq x}\left[\frac{x}{pq}\right] = \sum_{pq\leq x}\frac{x}{pq} + O\left(\sum_{pq\leq x} 1\right) = \sum_{pq\leq x}\frac{x}{pq} + O\left(\sum_{p\leq x}\pi\left(\frac{x}{p}\right)\right)$$
$$= x\sum_{pq\leq x}\frac{1}{pq} + O\left(x\sum_{p\leq x}\frac{1}{p}\right) = x\sum_{pq\leq x}\frac{1}{pq} + O\left(x\log\log x\right).$$

Note that we have

$$\left(\sum_{p\leq x^{1/2}}\frac{1}{p}\right)^2 \leq \sum_{pq\leq x}\frac{1}{pq} \leq \left(\sum_{p\leq x}\frac{1}{p}\right)^2,$$

$$\sum_{p\leq x^{1/2}}\frac{1}{p} = \log\log x^{1/2} + O(1) = \log\log x + O(1),$$

and

$$(\log\log x + O(1))^2 = (\log\log x)^2 + O(\log\log x).$$

So we get

$$\sum_{pq\le x}\frac{1}{pq} = (\log\log x)^2 + O(\log\log x),$$

and therefore

$$\sum_{n\le x}\omega(n)^2 = \sum_{n\le x}\omega(n)(\omega(n)-1) + \sum_{n\le x}\omega(n) = x(\log\log x)^2 + O(x\log\log x).$$

This concludes the proof. □

THEOREM 4.10. *For any $\varepsilon > 0$, the number of $n \in [1,x]$ such that*

$$|\omega(n) - \log\log n| > (\log\log n)^{1/2+\varepsilon}$$

is $o(x)$.

PROOF. If $n \le x^{1/e}$, then the number of such n is $o(x)$. If $n \in (x^{1/e}, x]$, then

$$\log\log x - 1 < \log\log n \le \log\log x,$$

so we only need to show that the number of $n \in (x^{1/e}, x]$ such that

$$|\omega(n) - \log\log x| > (\log\log x)^{1/2+\varepsilon}$$

is $o(x)$. By Theorems 4.8 and 4.9 we have

$$\begin{aligned}\sum_{n\le x}(\omega(n)-\log\log x)^2 &= \sum_{n\le x}\omega(n)^2 - 2\log\log x\sum_{n\le x}\omega(n) + (\log\log x)^2\sum_{n\le x}1\\ &= x(\log\log x)^2 + O(x\log\log x) - 2\log\log x(x\log\log x + O(x))\\ &\qquad + (\log\log x)^2(x+O(1))\\ &= O(x\log\log x).\end{aligned}$$

Let M denote the number of $n \in [1,x]$ such that $|\omega(n) - \log\log x| > (\log\log x)^{1/2+\varepsilon}$ is M, then we have

$$M\cdot(\log\log x)^{1+2\varepsilon} \le \sum_{n\le x}(\omega(n)-\log\log x)^2 \ll x\log\log x,$$

and therefore

$$M \ll \frac{x}{(\log\log x)^{2\varepsilon}} = o(x).$$

This proves the theorem. □

Note that $\Omega(n) \ge \omega(n)$ for all $n \ge 1$ and

$$\sum_{n\le x}(\Omega(n)-\omega(n)) = O(x).$$

So the number of $n \in [1,x]$ such that $\Omega(n) - \omega(n) > (\log\log x)^{1/2}$ is $O(\frac{x}{(\log\log x)^{1/2}})$. By Theorem 4.10, we have $\omega(n) \sim \Omega(n) \sim \log\log n$ for almost all $n \ge 1$.

4.3. Bertrand's Postulate

In 1845 J. Bertrand showed empirically that there is a prime between n and $2n$ for all n greater than 1 and less than six million, and predicted that this is true for all positive integers n. Chebyshev proved this in 1852.

THEOREM 4.11. *For any $n \geq 1$, there is at least one prime in $(n, 2n]$.*

REMARK 4.12. Theorem 4.5 implies a weak form of Theorem 4.11: there exists a positive constant c such that there is a prime between n and cn for all n. By Theorem 4.5 there is a constant A such that

$$\log n - A < \sum_{p \leq n} \frac{\log p}{p} < \log n + A$$

for all n. Hence

$$\sum_{n < p \leq e^{2A} n} \frac{\log p}{p} = \sum_{p \leq e^{2A} n} \frac{\log p}{p} - \sum_{p \leq n} \frac{\log p}{p} > \log e^{2A} n - A - \log n - A = 0.$$

So we can take $c = e^{2A}$.

REMARK 4.13. One can consider primes in short intervals $[x, x+y]$ with $y < x$ and $x \to \infty$. It it natural to ask whether there is a prime in $[n^2, (n+1)^2]$ for all large n. This is an open problem, known as Legendre's conjecture. It follows from a result by Ingham that for all sufficiently large n, there is a prime between the consecutive cubes n^3 and $(n+1)^3$. Baker, Harman and Pintz proved that there is a prime in the interval $[x - x^{0.525}, x]$ for all sufficiently large x.

To prove Theorem 4.11 we need two lemmas.

LEMMA 4.14. *For every positive integer n, we have $\prod_{p \leq n} p < 4^n$.*

PROOF. We use induction on n. If $n = 1$ or 2, the inequality is obvious. Suppose it is true for $1, 2, \ldots, n-1$, where $n \geq 3$. The we only need to consider odd n, since if n is even then

$$\prod_{p \leq n} p = \prod_{p \leq n-1} p < 4^{n-1} < 4^n.$$

Take $n = 2m+1$. Note that the binomial coefficient

$$\binom{2m+1}{m} = \frac{(2m+1)!}{m!(m+1)!}$$

is divisible by every prime p with $m+2 \leq p \leq 2m+1$. Hence we have

$$\prod_{p \leq 2m+1} p \leq \binom{2m+1}{m} \prod_{p \leq m+1} p \leq \binom{2m+1}{m} 4^{m+1}.$$

Note that $\binom{2m+1}{m} = \binom{2m+1}{m+1}$ are both occur in the expansion of $(1+1)^{2m+1}$, so that

$$\binom{2m+1}{m} \leq 2^{2m+1-1} = 4^m.$$

Hence

$$\prod_{p \leq 2m+1} p \leq 4^m 4^{m+1} = 4^{2m+1}.$$

The lemma follows by induction on n. $\square$

LEMMA 4.15. *If* $n \geq 3$ *and* $\frac{2}{3}n < p \leq n$*, then* $p \nmid \binom{2n}{n}$.

PROOF. By Theorem 1.15 we know if $p^{e(p,n)} \parallel \binom{2n}{n}$, then

$$e(p,n) = \sum_{k \geq 1} \left(\left[\frac{2n}{p^k}\right] - 2\left[\frac{n}{p^k}\right] \right).$$

Since $n \geq 3$ and $p > \frac{2}{3}n$, we have $p \geq 3$ and $p^2 > \frac{2p}{3}n \geq 2n$. By $\frac{2}{3}n < p \leq n$ we have $2 \leq \frac{2n}{p} < 3$ and $1 \leq \frac{n}{p} < \frac{3}{2}$. Hence for $n \geq 3$ and $\frac{2}{3}n < p \leq n$, we have

$$e(p,n) = \left[\frac{2n}{p}\right] - 2\left[\frac{n}{p}\right] = 2 - 2 \cdot 1 = 0.$$

This completes the proof. $\square$

PROOF OF THEOREM 4.11. There is such a prime for $n = 1$ or 2. Assume there is none for a certain integer $n \geq 3$. Hence

$$\binom{2n}{n} = \prod_{p \leq 2n} p^{e(p,n)} = \prod_{p \leq 2n} p^{e(p,n)}.$$

By Theorem 1.15 we have

$$e(p,n) = \sum_{k \geq 1} \left(\left[\frac{2n}{p^k}\right] - 2\left[\frac{n}{p^k}\right] \right).$$

Note that for $p^k \leq 2n$ we have $\left[\frac{2n}{p^k}\right] - 2\left[\frac{n}{p^k}\right] \leq 1$. So $p^{e(p,n)} \leq 2n$. If $\frac{2}{3}n < p \leq n$, by Lemma 4.15 we have $e(p,n) = 0$. If $\sqrt{2n} < p \leq \frac{2}{3}n$, then $e(p,n) \leq 1$. Hence

$$\binom{2n}{n} \leq \prod_{p \leq \sqrt{2n}} 2n \prod_{\sqrt{2n} < p \leq \frac{2}{3}n} p \leq \prod_{p \leq \sqrt{2n}} 2n \prod_{p \leq \frac{2}{3}n} p.$$

By Lemma 4.14 we get

$$\binom{2n}{n} \leq (2n)^{\pi(2n)} 4^{\frac{2}{3}n}.$$

But $\binom{2n}{n}$ is the largest of the $2n+1$ terms in the expansion of $(1+1)^{2n}$, and the first and last terms are 1, so that

$$2n\binom{2n}{n} > 4^n.$$

Note that $\pi(\sqrt{2n}) \leq \sqrt{2n} - 1$. We have

$$\frac{4^n}{2n} < (2n)^{\sqrt{2n}-1} 4^{\frac{2}{3}n}, \quad \text{and} \quad 4^{\frac{n}{3}} < (2n)^{\sqrt{2n}}.$$

Taking logarithms, we have

$$\frac{\log 4}{3} n < \sqrt{2n} \log 2n, \quad \text{and} \quad \sqrt{n} < \frac{3\sqrt{2}}{\log 4} \log 2n.$$

The inequality is false for $n \geq 427$. Indeed, let $f(n) = \sqrt{n} - \frac{3\sqrt{2}}{\log 4}\log 2n$. Then $f'(n) = \frac{1}{2\sqrt{n}} - \frac{3\sqrt{2}}{n(\log 4)} > 0$ if $n > 38$. Hence $f(n) \geq f(427) > 0$ if $n \geq 427$. So there is a prime between n and $2n$ for $n \geq 427$. But in the sequence of primes

$$2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 557,$$

each number is smaller than twice the one preceding it. So there is also such a prime for all $n \leq 427$, and hence for all $n \geq 1$. □

4.4. Averages of $\mu(n)$

THEOREM 4.16. *For all $n \in \mathbb{N}$, we have*

$$\sum_{m=1}^{n} \mu(m) \left[\frac{n}{m}\right] = 1.$$

Moreover for all real number $x \geq 1$, we have

$$\left| \sum_{1\leq m\leq x} \frac{\mu(m)}{m} \right| \leq 1.$$

PROOF. We have

$$\sum_{m=1}^{n} \mu(m) \left[\frac{n}{m}\right] = \sum_{m\leq n} \mu(m) \sum_{d\leq \frac{n}{m}} 1 = \sum_{1\leq \ell\leq n} \sum_{m|\ell} \mu(m).$$

By Theorem 2.18 we have $\sum_{m=1}^{n} \mu(m) \left[\frac{n}{m}\right] = \sum_{1\leq \ell \leq n} I(\ell) = 1$.

Denote $n = [x]$. Then we have

$$\begin{aligned}
\sum_{1\leq m\leq x} \frac{\mu(m)}{m} &= \sum_{1\leq m\leq n} \frac{\mu(m)}{m} = \frac{1}{n} \sum_{1\leq m\leq n} \mu(m)\frac{n}{m} \\
&= \frac{1}{n} \sum_{1\leq m\leq n} \mu(m) \left[\frac{n}{m}\right] + \frac{1}{n} \sum_{1\leq m\leq n} \mu(m) \left\{\frac{n}{m}\right\} \\
&\leq \frac{1}{n} + \frac{1}{n}(n-1) = 1,
\end{aligned}$$

if $n \geq 4$, as $\mu(4) = 0$ and $|\left\{\frac{n}{m}\right\}| \leq 1$. This proves theorem for $x \geq 4$. But we have $\sum_{1\leq m\leq x} \frac{\mu(m)}{m} = 1$ for $1 \leq x < 2$, $\sum_{1\leq m\leq x} \frac{\mu(m)}{m} = 1 - \frac{1}{2} = \frac{1}{2}$ for $2 \leq x < 3$, and $\sum_{1\leq m\leq x} \frac{\mu(m)}{m} = 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ for $3 \leq x < 4$. This completes the proof. □

DEFINITION 4.17. If $x \geq 1$ we define

$$M(x) := \sum_{n\leq x} \mu(n) \quad \text{and} \quad H(x) := \sum_{n\leq x} \mu(n)\log n.$$

THEOREM 4.18. *The following two assertions are equivalent:*

$$M(x) = o(x), \quad x \to \infty, \tag{4.5}$$

$$H(x) = o(x\log x), \quad x \to \infty. \tag{4.6}$$

Proof. By Theorem 3.5 with $\lambda_n = 1$, $c_n = \mu(n)$, and $f(n) = \log n$, we have

$$H(x) = M(x)\log x - \int_1^x \frac{M(t)}{t}\mathrm{d}t.$$

Note that $M(t) \le t$. We have

$$H(x) = M(x)\log x + O(x),$$

from which we know (4.5) $\Longleftrightarrow$ (4.6). □

Theorem 4.19. *The prime number theorem is equivalent to* (4.5).

Proof. The prime number theorem is equivalent to $\psi(x) \sim x$. Assume $\psi(x)-x = O(xR(x))$, for some $R(x)$ such that $R(x) \ge (\log x)^{-1}$ and $R(x) \to 0$ as $x \to \infty$. By $\Lambda = -\mu\log * u$, we have $\mu\log = -\mu * \Lambda$. Let $a = x/\log x$. Then by Theorems 3.14 and 4.16 we have

$$\begin{aligned}
H(x) &= \sum_{n\le x}\mu(n)\log n = -\sum_{n\le x}(\mu*\Lambda)(n)\\
&= -\sum_{n\le a}\mu(n)\sum_{m\le x/n}\Lambda(m) - \sum_{m\le x/a}\Lambda(m)\sum_{n\le x/m}\mu(n) + \sum_{n\le a}\mu(n)\sum_{m\le x/a}\Lambda(m)\\
&= -\sum_{n\le a}\mu(n)\frac{x}{n}(1+O(R(x/n))) + O\left(x\sum_{m\le x/a}\frac{\Lambda(m)}{m}\right) + O(x)\\
&= O\left(x\sum_{n\le a}\frac{R(x/n)}{n} + x\sum_{m\le x/a}\frac{\Lambda(m)}{m}\right) + O(x)\\
&= O\left(x\log x \max_{\log x\le y\le x} R(y) + x\log\log x\right) + O(x) = o(x\log x).
\end{aligned}$$

By Theorem 4.18 we know the prime number theorem implies (4.5).

Now we prove (4.5) implies the prime number theorem. We first show that

$$\psi(x) = x - \sum_{\substack{m \\ mn\le x}}\sum_{n}\mu(m)f(n) + O(1), \tag{4.7}$$

where

$$f(n) = \tau(n) - \log n - 2\gamma,$$

with γ the Euler constant. Note that

$$\sum_{n\le x} 1 = [x], \quad \sum_{n\le x}\Lambda(n) = \psi(x), \quad \sum_{n\le x} I(n) = 1,$$

and $u = \mu * \tau$, $\Lambda = \mu * \log$, and $I = \mu * u$. We have

$$\begin{aligned}
[x] - \psi(x) - 2\gamma &= \sum_{n\le x}(1 - \Lambda(n) - 2\gamma I(n))\\
&= \sum_{m\le x}\mu(m)\sum_{n\le x/m}(\tau(n) - \log n - 2\gamma) = \sum_{\substack{m \\ mn\le x}}\sum_{n}\mu(m)f(n).
\end{aligned}$$

This proves (4.7). So now we need to show that (4.5) implies

$$\sum_{\substack{m \\ mn\le x}}\sum_{n} \mu(m)f(n) = o(x). \tag{4.8}$$

Assume that $M(x) = O(xR(x))$ for some $R(x)$ such that $R(x) \ge (\log x)^{-1}$ and $R(x) \to 0$ as $x \to \infty$. Let $b = b(x) := (\max_{x^{1/2}\le y\le x} R(y))^{-1}$. We have $b \to \infty$ as $x \to \infty$. By Theorem 3.14, we get

$$\sum_{\substack{m \\ mn\le x}}\sum_{n} \mu(m)f(n) = \sum_{n\le b} f(n)M\left(\frac{x}{n}\right) + \sum_{n\le x/b} \mu(n)F\left(\frac{x}{n}\right) - F(b)M\left(\frac{x}{b}\right),$$

where $F(y) = \sum_{n\le y} f(n)$. By Theorems 3.3 and 3.15 we have

$$F(y) = \sum_{n\le y}\tau(n) - \sum_{n\le y}\log n - 2\gamma[y] = O(y^{1/2}).$$

Note that

$$\sum_{n\le b}\frac{\tau(n)}{n} \ll (\log b)^2, \quad \sum_{n\le b}\frac{\log n}{n} \ll (\log b)^2.$$

Hence we have

$$\begin{aligned}\sum_{n\le b} f(n)M\left(\frac{x}{n}\right) &\ll \sum_{n\le b}\frac{|f(n)|}{n} x \max_{x/b\le y\le x} R(y)\\ &\ll x(\log b)^2 \max_{x/b\le y\le x} R(y) \ll x\frac{(\log b)^2}{b} = o(x).\end{aligned}$$

and

$$\sum_{n\le x/b}\mu(n)F\left(\frac{x}{n}\right) \ll \sum_{n\le x/b}\sqrt{\frac{x}{n}} \ll \frac{x}{\sqrt{b}}, \quad F(b)M\left(\frac{x}{b}\right) \ll \frac{x}{\sqrt{b}}.$$

Hence we prove (4.8), and therefore we show (4.5) implies the prime number theorem. This concludes the proof. □

4.5. Average of $\lambda(n)$

DEFINITION 4.20. If $x \ge 1$ we define

$$L(x) := \sum_{n\le x}\lambda(n).$$

THEOREM 4.21. *The prime number theorem is equivalent to*

$$L(x) = o(x), \quad x \to \infty. \tag{4.9}$$

PROOF. By Theorem 4.19 we only need to prove $L(x) = o(x) \iff M(x) = o(x)$.

We first prove $M(x) = o(x) \implies L(x) = o(x)$. By Theorem 2.41, we have $\lambda = \mu * \mathbb{1}_\square$. Hence by Theorem 3.14 we have

$$L(x) = \sum_{n\le a}\mathbb{1}_\square(n)M\left(\frac{x}{n}\right) + \sum_{m\le x/a}\mu(m)G\left(\frac{x}{m}\right) - G(a)M\left(\frac{x}{a}\right),$$

where $G(y) = \sum_{n\le y} \mathbb{1}_\Box(n) \ll \sqrt{y}$. Take $a = x^{1/2}$. Assume $M(x) = o(x)$. Then we have

$$L(x) = \sum_{k\le x^{1/4}} \frac{o(x)}{k^2} + \sum_{m\le x^{1/2}} \sqrt{\frac{x}{m}} + x^{3/4} = o(x).$$

Now we show $L(x) = o(x) \Longrightarrow M(x) = o(x)$. Since $\lambda = \mu * \mathbb{1}_\Box$, we have $\mu = \lambda * \mu\mathbb{1}_\Box$. Hence by Theorem 3.14 we have

$$M(x) = \sum_{n\le a} \mu(n)\mathbb{1}_\Box(n) L\left(\frac{x}{n}\right) + \sum_{m\le x/a} \lambda(m) S\left(\frac{x}{m}\right) - S(a)L\left(\frac{x}{a}\right),$$

where $S(y) = \sum_{n\le y} \mu(n)\mathbb{1}_\Box(n) \ll \sqrt{y}$. Take $a = x^{1/2}$. Assume $L(x) = o(x)$. Then we have

$$M(x) = \sum_{k\le x^{1/4}} \frac{o(x)}{k^2} + \sum_{m\le x^{1/2}} \sqrt{\frac{x}{m}} + x^{3/4} = o(x).$$

This completes the proof. □

4.6. Selberg's formula

The prime number theorem was proved in 1896 independently by Hadamard and de la Valleé Poussin by analytic methods. The first elementary proofs were found about fifty years later by Erdös and Selberg based on Selberg's formula.

Theorem 4.22. *We have*

$$\sum_{n\le x} \Lambda(n)\log n + \sum\sum_{mn\le x} \Lambda(m)\Lambda(n) = 2x\log x + O(x).$$

Proof. By Theorem 2.48 we have $\Lambda_2 = \mu * (\log)^2 = \Lambda \log + \Lambda * \Lambda$. By the partial summation formula we have

$$\begin{aligned}\sum_{n\le x}(\log n)^2 &= \int_1^x (\log u)^2 \mathrm{d}\sum_{n\le u} 1 = [x](\log x)^2 - \int_1^x (u + O(1)) 2\log u \,\frac{\mathrm{d}u}{u} \\ &= x(\log x)^2 - 2x\log x + 2x + O((\log x)^2) \\ &= x\log x \left(\sum_{k\le x} \frac{1}{k}\right) - \sum_{k\le x}(\gamma + (\gamma+2)\log k) + O((\log x)^2),\end{aligned}$$

where γ is the Euler constant. Here we have used Theorems 3.3 and 3.4. Note that $\sum_{m\le x}(\log x/m)^2 = O(x)$. Hence we have

$$\begin{aligned}
\sum_{n\le x}\Lambda_2(n) &= \sum_{n\le x}\Lambda(n)\log n + \sum\sum_{mn\le x}\Lambda(m)\Lambda(n)\\
&= \sum_{m\le x}\mu(m)\sum_{n\le x/m}(\log n)^2\\
&= \sum_{m\le x}\mu(m)\frac{x}{m}\log\frac{x}{m}\left(\sum_{k\le\frac{x}{m}}\frac{1}{k}\right) - \sum_{m\le x}\mu(m)\sum_{k\le\frac{x}{m}}(\gamma+(\gamma+2)\log k) + O(x)\\
&= \sum_{\ell\le x}\frac{x}{\ell}\sum_{m|\ell}\mu(m)\log\frac{x}{m} - \sum_{\ell\le x}\sum_{m|\ell}\mu(m)(\gamma+(\gamma+2)\log\frac{\ell}{m}) + O(x).
\end{aligned}$$

By Theorems 2.18, 4.2 and 4.5 we have

$$\begin{aligned}
\sum_{n\le x}\Lambda_2(n) &= \sum_{\ell\le x}\frac{x}{\ell}(I(\ell)\log x + \Lambda(\ell)) - \sum_{\ell\le x}(\gamma I(\ell) + (\gamma+2)\Lambda(\ell)) + O(x)\\
&= 2x\log x + O(x).
\end{aligned}$$

This completes the proof. □

4.7. Elementary proof of the prime number theorem

In this section we will prove the prime number theorem.

THEOREM 4.23. *For $x \ge 2$ and $A > 0$ we have*

$$\psi(x) = x + O_A\left(\frac{x}{(\log x)^A}\right), \quad x\to\infty.$$

We shall derive Theorem 4.23 from a similar estimate for the sum of the Möbius function.

THEOREM 4.24. *For $x \ge 2$ and $A > 0$ we have*

$$M(x) = O_A\left(\frac{x}{(\log x)^A}\right), \quad x\to\infty.$$

PROOF OF THEOREM 4.23 BY ASSUMING THEOREM 4.24. The proof is similar to the proof of Theorem 4.19. We shall give some details for completeness. Recall that we have (4.7), that is,

$$\psi(x) = x - \sum_{m}\sum_{\substack{n\\ mn\le x}}\mu(m)f(n) + O(1),$$

where

$$f(n) = \tau(n) - \log n - 2\gamma,$$

with γ the Euler constant. So now we need to show that Theorem 4.24 implies

$$\sum_{m}\sum_{\substack{n\\ mn\le x}}\mu(m)f(n) = O_A\left(\frac{x}{(\log x)^A}\right). \tag{4.10}$$

Let $b = b(x) := (\log x)^{2A}$. By Theorem 3.14, we get

$$\sum_{\substack{m \\ mn \le x}} \sum_{n} \mu(m) f(n) = \sum_{n \le b} f(n) M\left(\frac{x}{n}\right) + \sum_{n \le x/b} \mu(n) F\left(\frac{x}{n}\right) - F(b) M\left(\frac{x}{b}\right),$$

where $F(y) = \sum_{n \le y} f(n)$. By Theorems 3.3 and 3.15 we have

$$F(y) = \sum_{n \le y} \tau(n) - \sum_{n \le y} \log n - 2\gamma[y] = O(y^{1/2}).$$

Note that

$$\sum_{n \le b} \frac{\tau(n)}{n} \ll (\log b)^2, \quad \sum_{n \le b} \frac{\log n}{n} \ll (\log b)^2.$$

Hence we have

$$\begin{aligned} \sum_{n \le b} f(n) M\left(\frac{x}{n}\right) &\ll \sum_{n \le b} \frac{|f(n)|}{n} \frac{x}{(\log x)^{A+1}} \\ &\ll (\log b)^2 \frac{x}{(\log x)^{A+1}} \ll \frac{x}{(\log x)^A}. \end{aligned}$$

and

$$\sum_{n \le x/b} \mu(n) F\left(\frac{x}{n}\right) \ll \frac{x}{\sqrt{b}} \ll \frac{x}{(\log x)^A}, \qquad F(b) M\left(\frac{x}{b}\right) \ll \frac{x}{\sqrt{b}} \ll \frac{x}{(\log x)^A}.$$

Hence we prove (4.10). This concludes the proof. □

To prove Theorem 4.24 we will need the following lemmas.

Lemma 4.25. *For* $\mathrm{Re}(s) > 1$ *we have*

$$\zeta^{(\ell)}(s) = \frac{(-1)^\ell \ell!}{(s-1)^{\ell+1}} + O_\ell((\log 2|s|)^{\ell+1}).$$

Proof. By the Euler–Maclaurin formula we derive (with any $X \ge 2$)

$$\begin{aligned} (-1)^\ell \zeta^{(\ell)}(s) &= \sum_{n=1}^{\infty} \frac{(\log n)^\ell}{n^s} \\ &= \sum_{n \le X} \frac{(\log n)^\ell}{n^s} + \int_X^\infty \frac{(\log y)^\ell}{y^s} \mathrm{d}y \\ &\quad + O\left(\int_X^\infty \left(|s| \frac{(\log y)^{\ell+1}}{y^{\mathrm{Re}(s)+1}} + \ell \frac{(\log y)^{\ell-1}}{y^{\mathrm{Re}(s)+1}}\right) \mathrm{d}y + \frac{(\log X)^\ell}{X^{\mathrm{Re}(s)}}\right) \\ &= \sum_{n \le X} \frac{(\log n)^\ell}{n^s} + \int_X^\infty \frac{(\log y)^\ell}{y^s} \mathrm{d}y + O\left(\frac{|s|}{X} (\log X)^{\ell+1}\right). \end{aligned}$$

Here we have used the fact that for $\mathrm{Re}(s) > 1$

$$\begin{aligned}\int_X^\infty \frac{(\log y)^{\ell+1}}{y^{\mathrm{Re}(s)+1}}\mathrm{d}y &= \frac{1}{-\mathrm{Re}(s)}\int_X^\infty (\log y)^{\ell+1}\mathrm{d}y^{-\mathrm{Re}(s)}\\ &= \frac{1}{-\mathrm{Re}(s)}(\log y)^{\ell+1}y^{-\mathrm{Re}(s)}\Big|_X^\infty + \frac{\ell+1}{\mathrm{Re}(s)}\int_X^\infty y^{-\mathrm{Re}(s)-1}(\log y)^\ell \mathrm{d}y\\ &= O\left(\frac{1}{X}(\log X)^{\ell+1}\right).\end{aligned}$$

Note that

$$\sum_{n\le X}\frac{(\log n)^\ell}{n^s} \le \sum_{n\le X}\frac{(\log n)^\ell}{n^{\mathrm{Re}\, s}} \le \sum_{n\le X}\frac{(\log n)^\ell}{n} \ll (\log X)^{\ell+1}$$

and

$$\int_1^X \frac{(\log y)^\ell}{y^s}\mathrm{d}y \le \int_1^X \frac{(\log y)^\ell}{y}\mathrm{d}y \ll (\log X)^{\ell+1}.$$

Hence

$$(-1)^\ell\zeta^{(\ell)}(s) = \int_1^\infty \frac{(\log y)^\ell}{y^s}\mathrm{d}y + O\left(\left(1+\frac{|s|}{X}\right)(\log X)^{\ell+1}\right).$$

Note that

$$\int_1^\infty \frac{(\log y)^\ell}{y^s}\mathrm{d}y = \int_0^\infty x^\ell e^{-(s-1)x}\mathrm{d}x = \int_0^\infty (s-1)^{-\ell}u^\ell e^{-u}\mathrm{d}u = \Gamma(\ell+1)(s-1)^{-\ell}$$

and $\Gamma(\ell+1) = \ell!$. By taking $X = 2|s|$, we complete the proof. □

LEMMA 4.26. *For Re$(s) > 1$ we have*

$$\Big((s-1)\zeta(s)\Big)^{(\ell)} \ll |s|(\log 2|s|)^{\ell+1}.$$

PROOF. By induction we have

$$\Big((s-1)\zeta(s)\Big)^{(\ell)} = (s-1)\zeta^{(\ell)}(s) + \ell\zeta^{(\ell-1)}(s).$$

The lemma follows from Lemma 4.25 by the above formula. □

PROOF OF THEOREM 4.24. First we are going to estimate the series

$$G(s) = \sum_{n=1}^\infty \frac{\mu(m)}{m^s}(\log m)^k = (-1)^k\left(\frac{1}{\zeta(s)}\right)^{(k)}$$

for $k \ge 0$ and $s = \sigma + it$, $\sigma > 1$ and $t \in \mathbb{R}$. Put $\zeta^*(s) = (s-1)\zeta(s)$. We need a lower bound for $\zeta^*(s)$. To do this we use the Euler product for $\zeta(s)$ giving

$$\begin{aligned}1 &\le \prod_p\left(1+\frac{(1+p^{it}+p^{-it})^2}{p^\sigma}\right)\\ &= \prod_p\left(1+\frac{3+2p^{it}+2p^{-it}+p^{2it}+p^{-2it}}{p^\sigma}\right)\\ &\asymp \zeta(\sigma)^3|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)|^2.\end{aligned}$$

If $|s-1| \geq \epsilon$, then we have $|\zeta(\sigma+2it)| \ll \log 2|s|$ by Lemma 4.25 with $\ell = 0$. So we have

$$|\zeta(\sigma+it)| \gg \frac{1}{((\frac{1}{(\sigma-1)})^3(\log 2|s|)^2)^{1/4}} = (\sigma-1)^{3/4}(\log 2|s|)^{-1/2},$$

and hence

$$|\zeta^*(s)| \gg (\sigma-1)^{3/4}|s|(\log 2|s|)^{-1/2}. \tag{4.11}$$

If $|s-1| \leq \epsilon$, then we have $|\zeta^*(s)| \asymp 1$ by Lemma 4.25 with $\ell = 0$, and hence (4.11) also holds.

By induction, we have

$$(-1)^k G(s) = \left(\frac{s-1}{\zeta^*(s)}\right)^{(k)} = (s-1)\left(\frac{1}{\zeta^*(s)}\right)^{(k)} + k\left(\frac{1}{\zeta^*(s)}\right)^{(k-1)}.$$

By the formula from the differential calculus

$$\left(\frac{1}{f}\right)^{(k)} = \frac{k!}{f}\sum_{a_1+2a_2+\cdots+ka_k=k} \frac{(a_1+a_2+\cdots+a_k)}{a_1!a_2!\cdots a_k!}\left(\frac{-f'}{1!f}\right)^{a_1}\left(\frac{-f''}{2!f}\right)^{a_2}\cdots\left(\frac{-f^{(k)}}{k!f}\right)^{a_k}$$

with $f = \zeta^*$ we get

$$\left(\frac{1}{\zeta^*(s)}\right)^{(k)} \ll (\sigma-1)^{-\frac{3}{4}(k+1)}\frac{(\log 2|s|)^\kappa}{|s|},$$

for some κ depending on k, for example $\kappa = \frac{1}{2}(5k+1)$. Hence

$$G(s) \ll_k (\sigma-1)^{-\frac{3}{4}(k+1)}(\log 2|s|)^\kappa. \tag{4.12}$$

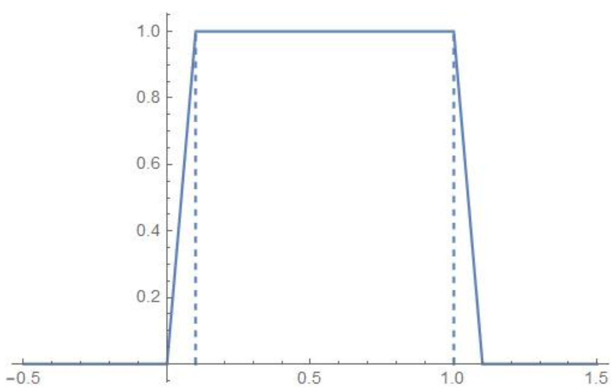

FIGURE 1.

Next we derive an estimate for the finite sum

$$F(x) = \sum_{m\leq x}\frac{\mu(m)}{m^\sigma}(\log m)^k, \qquad x>1,\ k\geq 1$$

from the estimate (4.12). We first smooth out at the endpoint of summation by means of the function

$$\Delta(y) = \begin{cases} 1, & \delta < x < 1, \\ 0, & x<0 \text{ or } x>1+\delta, \\ \delta^{-1}x, & 0\leq x\leq \delta, \\ -\delta^{-1}x+\delta^{-1}+1, & 1\leq x\leq 1+\delta, \end{cases}$$

whose graph is given by Figure 1 with $0 < \delta < 1$ to be chosen later. We have

$$F(x) = \sum_{m\geq 1} \frac{\mu(m)}{m^\sigma}(\log m)^k \Delta\left(\frac{\log m}{\log x}\right) + O(\delta(\log x)^{k+1}).$$

Let $\hat{\Delta}$ be the Fourier transform of $\Delta(y)$, then we have

$$\begin{aligned}\hat{\Delta}(u) &= \int_{-\infty}^{\infty} \Delta(y)e(-yu)\mathrm{d}y = \frac{\sin(\pi u)\sin(\delta\pi u)}{\delta\pi^2 u^2} e^{i\pi(1+\delta)u} \\ &\ll \min\left\{1, \frac{1}{|u|}, \frac{1}{\delta|u|^2}\right\} \ll \frac{1}{1+|u|+\delta|u|^2}.\end{aligned}$$

By the Fourier inversion we have

$$\Delta(y) = \int_{-\infty}^{\infty} \hat{\Delta}(u)e(uy)\mathrm{d}u.$$

Hence we have

$$\begin{aligned}F(x) &= \sum_{m\geq 1} \frac{\mu(m)}{m^\sigma}(\log m)^k \int_{-\infty}^{\infty} \hat{\Delta}(u) e\left(u\frac{\log m}{\log x}\right)\mathrm{d}u + O(\delta(\log x)^{k+1}) \\ &= \int_{-\infty}^{\infty} \hat{\Delta}(u) G\left(\sigma - i\frac{2\pi u}{\log x}\right)\mathrm{d}u + O(\delta(\log x)^{k+1}) \\ &\ll_k (\sigma-1)^{-\frac{3}{4}(k+1)} \int_0^{\infty} \frac{\log^\kappa(2+|u|)}{1+|u|+\delta|u|^2}\mathrm{d}u + \delta(\log x)^{k+1}.\end{aligned}$$

Note that

$$\int_0^{\infty} \frac{\log^\kappa(2+|u|)}{1+|u|+\delta|u|^2}\mathrm{d}u \ll \int_0^{1/\delta} \frac{\log^\kappa(2+|u|)}{1+|u|}\mathrm{d}u + \int_{1/\delta}^{\infty} \frac{\log^\kappa(2+|u|)}{\delta|u|^2}\mathrm{d}u \ll \left(\log\frac{1}{\delta}\right)^{\kappa+1}.$$

Choosing $\delta = (\log x)^{-k-1}$ we obtain

$$F(x) \ll_k (\sigma-1)^{-\frac{3}{4}(k+1)} (\log\log 5x)^{\kappa+1}.$$

Finally, by partial summation formula we get

$$\begin{aligned}M(x) = \sum_{m\leq x} \mu(m) &= \int_1^x \frac{y^\sigma}{(\log y)^k}\mathrm{d}F(y) \\ &= \frac{y^\sigma}{(\log y)^k}F(y)\Big|_1^x - \int_1^x F(y)\mathrm{d}\frac{y^\sigma}{(\log y)^k} \\ &\ll_k \frac{x^\sigma}{(\log x)^k}(\sigma-1)^{-\frac{3}{4}(k+1)} (\log\log 5x)^{\kappa+1} \ll x(\log x)^{\frac{3}{4}-\frac{k}{4}} (\log\log 5x)^{\kappa+1},\end{aligned}$$

by taking $\sigma = 1 + \frac{1}{\log x}$. This completes the proof of Theorem 4.24. □