Homework 4

(due Wednesday, April 28, 2021)

1. Show that for $k \geq 2$ we have
$$\sum_{n \leq x} \Lambda_k(n) = kx(\log x)^{k-1}\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

2. Define the counting function $\pi_{\odot}(R)$ as the number of prime lattice points, both coordinates are prime (positive or negative), inside the circle with radius R and origin $O = (0,0)$. That is, $\pi_{\odot}(R) := \#\{(p,q) \in \mathbb{Z}^2 : p, q \text{ are prime and } p^2 + q^2 \leq R^2\}$. Show that
$$\pi_{\odot}(R) \sim \frac{\pi R^2}{(\log R)^2}, \quad R \to \infty.$$

3. Let $N_{\square}(R) = \#\{(m,n) \in \mathbb{Z}^2 : \gcd(m,n) = 1 \text{ and } |m| \leq R,\ |n| \leq R\}$. Show that
$$N_{\square}(R) = \frac{24}{\pi^2}R^2 + O(R \log R).$$

4. Let $N_{\odot}(R) = \#\{(m,n) \in \mathbb{Z}^2 : \gcd(m,n) = 1 \text{ and } m^2 + n^2 \leq R^2\}$. Show that
$$N_{\odot}(R) = \frac{6}{\pi}R^2 + O(R \log R).$$

5. If $m_1, m_2, \cdots, m_k$ are pairwise prime, and $x_1, x_2, \cdots, x_k$ run over a complete residue system modulo $m_1, m_2, \cdots, m_k$ respectively, then
$$x_1 + m_1x_2 + m_1m_2x_3 + \cdots + m_1m_2\cdots m_{k-1}x_k$$
run over a complete residue system modulo $m_1m_2\cdots m_k$.

6. Let $m > 1$ be an integer and $a \in \mathbb{N}$, $(a,m) = 1$. Show that
   (a) $\displaystyle\sum_{\xi \pmod m} \left\{\frac{a\xi+b}{m}\right\} = \frac{m-1}{2}$,
   (b) $\displaystyle\sideset{}{'}\sum_{\xi \pmod m} \left\{\frac{a\xi}{m}\right\} = \frac{1}{2}\varphi(m)$,

   where $\sum_{\xi \pmod m}$ means summing over a complete residue system modulo m, and $\sum'_{\xi \pmod m}$ means summing over a reduced residue system modulo m.

7. If p is an odd prime, show that not all three of the sets
$$\{a_1, \cdots, a_{p-1}\},\ \{b_1, \cdots, b_{p-1}\}, \{a_1b_1, \cdots, a_{p-1}b_{p-1}\}$$
can be reduced residue systems $\pmod p$.

8. Show that for $1 < k < p-1$, we have $(p-k)!(k-1)! \equiv (-1)^k \pmod p$.

9. Solve the system of congruences

$$3x \equiv 9 \pmod{12}, \quad 4x \equiv 5 \pmod{35}, \quad 6x \equiv 2 \pmod{11}.$$

10. Let $p > 3$ be a prime number. Show that
    (a) $\left(\frac{3}{p}\right) = 1$ if and only if $p = 12n \pm 1$;
    (b) $\left(\frac{-3}{p}\right) = 1$ if and only if $p = 6n + 1$.

11. Show that if p is an odd prime and $\mathrm{ord}_p(a) = t > 1$, then

$$\sum_{k=1}^{t-1} a^k \equiv -1 \pmod{p}.$$

12. Show that a primitive root of p^k is also a primitive root of p^j for $1 \leq j < k$.

13. Solve the congruences
    (a) $x^{12} \equiv 16 \pmod{17}$;
    (b) $7x^7 \equiv 11 \pmod{41}$.