Homework 5

(due Monday, May 17, 2021)

1. Let $U = (u_{kn})$ be the $q \times q$ matrix with elements $u_{kn} = e(kn/q)/\sqrt{q}$. Show that $UU^* = U^*U = I$, i.e., that U is unitary.

2. i) Show that if $q > 1$, then
$$\sum_{n=1}^{q} c_q(n) = 0.$$
ii) Show that if $q_1 \neq q_2$ and $[q_1, q_2] \mid N$, then
$$\sum_{n=1}^{N} c_{q_1}(n) c_{q_2}(n) = 0.$$
iii) Show that if $q \mid N$, then
$$\sum_{n=1}^{N} |c_q(n)|^2 = N\varphi(q).$$

3. Show that
$$\sum_{d \mid q} |c_d(n)| = 2^{\omega(q/(q,n))}(q, n).$$

4. Show that for arbitrary real or complex numbers $c_1, \ldots, c_q$,
$$\sum_{\chi \bmod q} \left| \sum_{n=1}^{q} c_n \chi(n) \right|^2 = \varphi(q) \sum_{\substack{n=1 \\ (n,q)=1}}^{q} |c_n|^2.$$

5. Show that for arbitrary real or complex numbers c_χ,
$$\sum_{n=1}^{q} \left| \sum_{\chi \bmod q} c_\chi \chi(n) \right|^2 = \varphi(q) \sum_{\chi \bmod q} |c_\chi|^2.$$