# Homework 6

(due Monday, May 31, 2021)

1. Suppose that $\chi$ is a non-principal Dirichlet character. Show that

$$\sum_{p\le x} \frac{\chi(p)}{p} = b(\chi) + O_\chi\left(\frac{1}{\log x}\right),$$

where

$$b(\chi) = \log L(1,\chi) - \sum_{\substack{p^k \\ k>1}} \frac{\chi(p)^k}{kp^k}.$$

2. Let $\chi$ be a Dirichlet character modulo $q$. Show that if $\Re(s) = \sigma > 1$, then

$$\sum_{n=1}^{\infty} (-1)^{n-1}\chi(n)n^{-s} = (1-\chi(2)2^{1-s})L(s,\chi),$$

and

$$\sum_{n=1}^{\infty} \tau(n)^2\chi(n)n^{-s} = \frac{L(s,\chi)^4}{L(2s,\chi^2)}.$$

3. Let $\varphi^*(q)$ be the number of primitive characters modulo $q$. Show that

   i) $\varphi^*$ is a multiplicative function.

   ii) $\sum_{d|q} \varphi^*(d) = \varphi(q)$.

   iii) $\varphi^*(q) = q\prod_{p\|q}(1-\frac{2}{p})\prod_{p^2|q}(1-\frac{1}{p})^2$.

   iv) $\varphi^*(q) > 0$ if and only if $q \not\equiv 2 \pmod 4$.

4. i) Show that

$$\frac{1}{\varphi(q)}\sum_{\chi\ (q)} \bar{\chi}(a)\tau(\chi) = \begin{cases} e(a/q), & (a,q)=1 \\ 0, & \text{otherwise.}\end{cases}$$

   ii) Show that for all integers $a$,

$$e(a/q) = \sum_{d|(q,a)} \frac{1}{\varphi(q/d)} \sum_{\chi\ (q/d)} \bar{\chi}(a/d)\tau(\chi).$$

5. Suppose that $\chi_i$ is a character modulo $q_i$ for $i = 1, 2$, with $(q_1, q_2) = 1$. Show that

$$c_{\chi_1\chi_2}(n) = c_{\chi_1}(n)c_{\chi_2}(n)\chi_1(q_2)\chi_2(q_1).$$