# Homework 8

(due Thursday, May 19, 2022)

1. Let $\chi$ be a Dirichlet character modulo $q$. Show that if $\Re(s) = \sigma > 1$, then
$$\sum_{n=1}^{\infty}(-1)^{n-1}\chi(n)n^{-s} = (1-\chi(2)2^{1-s})L(s,\chi),$$
and
$$\sum_{n=1}^{\infty}\tau(n)^2\chi(n)n^{-s} = \frac{L(s,\chi)^4}{L(2s,\chi^2)}.$$

2. Suppose that $\chi$ is a non-principal Dirichlet character. Show that
$$\sum_{p\le x}\frac{\chi(p)}{p} = b(\chi) + O_\chi\left(\frac{1}{\log x}\right),$$
where
$$b(\chi) = \log L(1,\chi) - \sum_{\substack{p^k \\ k>1}}\frac{\chi(p)^k}{kp^k}.$$

3. The Kloosterman sum $S(m,n;q)$ is defined as follows:
$$S(m,n;q) = \sum_{\substack{d=1 \\ (d,q)=1}}^{q} e\left(\frac{md+n\bar{d}}{q}\right),$$
where $\bar{d}$ is the multiplicative inverse of $d$ mod $q$. Here $e(z) = e^{2\pi iz}$. When $q \mid n$, this is the Ramanujan sum $c_q(m)$. Prove the following properties of Kloosterman sums:
   - i) $S(m,n;q) = S(n,m;q)$,
   - ii) $S(m,n;q) = S(1,mn;q)$ whenever $(m,q)=1$,
   - iii) $S(m,n;qr) = S(\bar{r}m,\bar{r}n;q)S(\bar{q}m,\bar{q}n;r)$ if $(q,r)=1$. Here $\bar{q},\bar{r}$ are multiplicative inverses of $q,r$ to moduli $r,q$ respectively.

4. Let $\chi_i$ be a Dirichlet character modulo $q_i$ for $i=1,2$, with $(q_1,q_2)=1$. Show that
$$c_{\chi_1\chi_2}(n) = c_{\chi_1}(n)c_{\chi_2}(n)\chi_1(q_2)\chi_2(q_1).$$

5. LeVeque [§7.1, Problem 2].

6. LeVeque [§7.2, Problem 3].

7. LeVeque [§7.2, Problem 4].