Shandong University | Advanced Modern Algebra

Lecture 1 & 2: Multilinear Algebra

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This lecture refers to §10.4, §11.2 and §11.5 in [1]. All the equation numbers without reference labels are from this book. For convenience, [2, Chap. VI] and [3, Chap. 9] define tensor product of vector spaces without primilinary in modules.

1 Tensor products of vector spaces

In this section we study the tensor product of two vector spaces V and W over a field F, ultimately denoting it by $V \otimes_F W$.

1.1 Change of base for vector spaces

Let K/F be a finite extension of fields, and let W be an n-dimensional vector space over K (i.e. $W \cong K^n$). Then W automatically has a multiplication-by-F structure and is a vector space over the smaller field F. In this situation we ignore some unnecessary scalar multiplications and denote the F-vector space as W_F.

Exercise. *What is the dimension of W_F? Can you determine a basis of it?*

Now we try to reverse this, and let V be an n-dimensional vector space over F. How to construct a vector space over the larger field K that is "small enough" in which we can embed V?

Example (Complexification of a real vector space)**.** *Let $F = \mathbb{R}$, $K = \mathbb{C}$. If V is a real vector space, to construct a complex vector space $V^{\mathbb{C}}$ from V (which is called the **complexification** of V), it is necessary (and sufficient) to formally define a real linear transformation called "multiplication by i". More precisely*

$$(V^{\mathbb{C}})_{\mathbb{R}} := V \oplus iV = \{v_1 + iv_2 \mid v_1, v_2 \in V\}$$

as real vector spaces. One can define multiplication by a complex number on $V \oplus iV$ by the distributive law.

If W is an n-dimensional complex vector space, it is natually a $2n$-dimensional real vector space and we denote it by $W_{\mathbb{R}}$. If it is related to another real vector space V such that $W_{\mathbb{R}} = V \oplus iV$, we say that V is a ***real form*** *of the complex vector space W. We have shown that any real vector space is a real form of its complexification.*

The operations $(\cdot)_{\mathbb{R}}$ and $(\cdot)^{\mathbb{C}}$ are not inverse to each other. (Try to compute the dimensions of $(V^{\mathbb{C}})_{\mathbb{R}}$ and $(W_{\mathbb{R}})^{\mathbb{C}}$.)

We begin the construction by returning to the basic axioms of vector spaces in order to examine whether we can define "scalar products" of the form λv, for $\lambda \in K$ and $v \in V$. These axioms start with an abelian group V together with a map from $K \times V$ to V, where the image of the pair (λ, v) is denoted by λv.

It is therefore natural to consider the free abelian group on the set $K \times V$, i.e. the collection of all finite commuting sums of elements of the form (λ_i, v_i) where $\lambda_i \in K$ and $v_i \in V$. (The set $K \times V$ can also be regarded as an F-vector space by taking the direct sum $K \oplus V$. Notice that the free abelian group $\langle K \times V \rangle$ and $K \oplus V$ have different definitions of additions.) In this abelian group the original space V has been thoroughly distinguished from the new "coefficients" from K.

To satisfy the relations necessary for a K-vector space structure and the compatibility relation with the action of F-multiplication on V, we must take the quotient of this abelian group by the subgroup H generated by all elements of the form

$$\begin{gathered}(\lambda_1 + \lambda_2, v) - (\lambda_1, v) - (\lambda_2, v), \\ (\lambda, v_1 + v_2) - (\lambda, v_1) - (\lambda, v_2), \quad \text{and} \\ (\lambda a, v) - (\lambda, av),\end{gathered} \tag{10.3}$$

for $\lambda, \lambda_1, \lambda_2 \in K$, $v, v_1, v_2 \in V$ and $a \in F$, where av in the last element refers to the F-multiplication structure already defined on V.

The resulting quotient group (quotient space) is denoted by $K \otimes_F V$ (or just $K \otimes V$ if F is clear from the context) and is called the **tensor product** of K and V over F. If $\lambda \otimes v$ denotes the coset containing (λ, v) in $K \otimes_F V$ then by definition of the quotient we have forced the relations

$$\begin{gathered}(\lambda_1 + \lambda_2) \otimes v = \lambda_1 \otimes v + \lambda_2 \otimes v, \\ \lambda \otimes (v_1 + v_2) = \lambda \otimes v_1 + \lambda \otimes v_2, \quad \text{and} \\ (\lambda a) \otimes v = \lambda \otimes av.\end{gathered} \tag{10.4}$$

The elements of $K \otimes_F V$ are called **tensors** and can be written (non-uniquely in general) as finite sums of "simple tensors" of the form $\lambda \otimes v$ with $\lambda \in K$, $v \in V$.

Proposition. *The tensor product $K \otimes_F V$ is natually a vector space over K with the scalar multiplication defined by*

$$\lambda \left(\sum_{\text{finite}} \lambda_i \otimes v_i \right) := \sum_{\text{finite}} (\lambda \lambda_i) \otimes v_i .$$

The vector space $K \otimes_F V$ is called the K-vector space obtained by **extension of scalars** from the F-space V. It is also denoted by V^K in [2].

Exercise ([1] §11.2 Proposition 15)**.** *What is $\dim_K(V^K)$? Determine a basis of it. And what is $\dim_F((V^K)_F)$?*

Proposition. *The map $\iota : v \mapsto (1, v) \mapsto 1 \otimes v$ defines a natural embedding (a 1-1 F-linear map) $V \to K \otimes_F V$, more precisely,*

$$\iota : V \to K \times V \twoheadrightarrow K \otimes_F V = (K \otimes_F V)_F.$$

(For an extension of scalars from an R-module N, this map $\iota : N \to S \otimes_R N$ is not injective in general.)

Exercise. *Show that $F \otimes_F V \cong V$.*

The above proposition shows that $K \otimes_F V$ contains (an isomorphic copy of) V. On the other hand, the relations in equation (10.3) were the minimal relations that we had to impose in order to obtain a K-vector space, so it is reasonable to expect that the tensor product $K \otimes_F V$ is the "best possible" K-vector space to serve as target for an F-linear map from V. The next theorem makes this more precise by showing that any other F-linear map from V to any K-vector space factors through this one, and is referred to as the **universal property** for the tensor product $K \otimes_F V$.

Theorem ([1] §10.4 Theorem 8, universal property)**.** *Let F be a subfield of K, let V be a vector space over F and let $\iota : V \to K \otimes_F V$ be the F-linear map defined by $\iota(v) := 1 \otimes v$. Suppose that W is any vector space over K (hence also an F-linear space) and that $\varphi : V \to W$ is an F-linear map. Then there is a unique K-linear map $\tilde{\varphi} : W \to K \otimes_F V$ such that φ factors through $\tilde{\varphi}$, i.e., $\varphi = \tilde{\varphi} \circ \iota$ and the diagram*

$$\begin{array}{ccc} V & \overset{\iota}{\hookrightarrow} & K \otimes_F V \\ & {\scriptstyle\varphi}\searrow & \vdots\ {\scriptstyle\tilde{\varphi}} \\ & & W \end{array}$$

commutes.

1.2 Tensor products of two vector spaces

Let V, W be vector spaces over the field F. The quotient of the free abelian group on the set $V \times W$ by the subgroup generated by all elements of the form

$$\begin{gathered} (v_1 + v_2, w) - (v_1, w) - (v_2, w), \\ (v, w_1 + w_2) - (v, w_1) - (v, w_2), \quad \text{and} \\ (av, w) - (v, aw), \end{gathered} \tag{10.6}$$

for $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$ and $a \in F$, is an abelian group, denoted by $V \otimes_F W$ (or simply $V \otimes W$ if the field F is clear from the context), and is called the **tensor product** of V and W over F.

We have the relations

$$\begin{gathered} (v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, \\ v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2, \quad \text{and} \\ (av) \otimes w = v \otimes (aw). \end{gathered} \tag{10.7}$$

The elements of $V \otimes_F W$ are called **tensors**, and the coset, $v \otimes w$, of (v, w) in $V \otimes_F W$ is called a **simple tensor** or a **pure tensor**. Not every element need be a simple tensor, but every tensor can be written (non-uniquely in general) as finite sums of simple tensors. By the last relation in (10.7), we may have $v \otimes w = v' \otimes w'$ even if $v \neq v'$, $w \neq w'$.

Exercise. *If $v_1, v_2 \in V$, $w_1, w_2 \in W$ are linearly independent, is it possible to write $v_1 \otimes w_1 + v_2 \otimes w_2$ as a pure tensor?*

Proposition. *The tensor product $V \otimes_F W$ is a vector space over F with the scalar multiplication defined by*

$$a \left(\sum_{\textit{finite}} v_i \otimes w_i \right) := \sum_{\textit{finite}} (av_i) \otimes w_i = \sum_{\textit{finite}} v_i \otimes (aw_i).$$

Proposition ([1] §11.2 Proposition 16, [2] Proposition 6.14)**.** *If V, W are finite-dimensional vector spaces over F, then*

$$\dim_F (V \otimes_F W) = \dim_F V \dim_F W.$$

If $\{e_i\}$ is a basis of V and $\{f_j\}$ is a basis of W, then the most general member of $V \otimes_F W$ is of the form $\sum_j v_j \otimes f_j$ with all $v_j \in V$. In particular, $\{e_i \otimes f_j\}$ is a basis of $V \otimes_F W$.

1.3 Dual of tensor products

Fix a field F, and let U, V, W be vector spaces over F. The space of all linear transformations $\mathcal{A} : V \to W$ is a vector space over F under addition and scalar multiplication of the values, and is denoted by $\mathrm{Hom}_F(V, W)$.

Recall ([1] §11.2)**.** *Let V, W be of finite dimensional. What are the dimensions of* $\mathrm{Hom}_F(V, W)$ *and the dual space $V^* = \mathrm{Hom}_F(V, F)$ of V? Determine bases of them.*

A map $\varphi : V \times W \to U$ is said to be **bilinear** if it is linear in each of the two variables when the other one is held fixed, i.e.,

$$\begin{aligned} \varphi(v_1 + v_2, w) &= \varphi(v_1, w) + \varphi(v_2, w), \\ \varphi(v, w_1 + w_2) &= \varphi(v, w_1) + \varphi(v, w_2), \\ \varphi(av, w) &= \varphi(v, aw) = a\varphi(v, w), \end{aligned}$$

for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$ and $a \in F$. Such a space of bilinear maps (or bilinear functions) is a vector space over F under addition and scalar multiplication of the values, and is denoted by $\mathcal{L}(V, W; U)$. The bilinear maps are called **bilinear forms** when the range space U is F itself.

With this teminology, it follows immediately from the relations in (10.7) that, the quotient map $\iota : V \times W \to V \otimes_F W$ defined by $\iota(v, w) := v \otimes w$ is bilinear.

Theorem ([1] §10.4 Theorem 10 & Corollary 12, universal property)**.** *Let V, W be vector spaces over the field F, $V \otimes_F W$ be the tensor product of V and W, and $\iota : V \times W \to V \otimes_F W$ be the bilinear map defined above.*

(1) If $\tilde{\varphi} : V \otimes_F W \to U$ is any linear transformation to a vector space U, then the composite map $\tilde{\varphi} \circ \iota$ is a bilinear map from $V \times W$ to U.

(2) Conversely, suppose that U is any vector space over F and that $\varphi : V \times W \to U$ is a bilinear map. Then there is a unique F-linear map $\tilde{\varphi} : V \otimes_F W \to U$ such that φ factors through $\tilde{\varphi}$, i.e., $\varphi = \tilde{\varphi} \circ \iota$.

Equivalently, the correspondence $\varphi \leftrightarrow \tilde{\varphi}$ *in the commutative diagram*

$$\begin{array}{ccc} V \times W & \xrightarrow{\iota} & V \otimes_F W \\ & \searrow^{\varphi} & \Big\downarrow{\tilde{\varphi}} \\ & & U \end{array}$$

establishes a bijection $\mathcal{L}(V, W; U) \leftrightarrow \operatorname{Hom}_F(V \otimes_F W, U)$. *Furthermore, this bijection is an isomorphism of vector spaces.*

Corollary ([2] Corollary 6.12, an equivalent definition of tensor product)**.** *If* V *and* W *are vector spaces over* F*, then the vector space* $\mathcal{L}(V, W; F)$ *of all bilinear forms on* $V \times W$ *is canonically isomorphic to* $(V \otimes_F W)^*$*, the dual of the vector space* $V \otimes_F W$.

1.4 Tensor products of linear transformations

Theorem ([1] §10.4 Theorem 13)**.** *Let* V, V', W, W' *be vector spaces over* F*, and suppose* $\mathcal{A} : V \to V'$*,* $\mathcal{B} : W \to W'$ *are* F*-linear maps.*

(1) There is a unique F*-linear map* $\mathcal{C} : V \otimes_F W \to V' \otimes_F W'$*, such that*

$$\mathcal{C}(v \otimes w) = \mathcal{A}v \otimes \mathcal{B}w, \quad \text{for all } v \in V,\ w \in W.$$

Denote $\mathcal{C}$ *by* $\mathcal{A} \otimes \mathcal{B}$.

(2) If $\mathcal{A}' : V' \to V''$*,* $\mathcal{B}' : W' \to W''$ *are also* F*-linear maps, then*

$$(\mathcal{A}' \otimes \mathcal{B}') \circ (\mathcal{A} \otimes \mathcal{B}) = (\mathcal{A}' \circ \mathcal{A}) \otimes (\mathcal{B}' \circ \mathcal{B}).$$

Proof. The map $(v, w) \mapsto \mathcal{A}v \otimes \mathcal{B}w$ from $V \times W$ to $V' \otimes_F W'$ is clearly bilinear, so (1) follows immediately from the universal property.

The uniqueness condition of the universal property also implies (2). □

Exercise. *Show that the above theorem constructs an injection from* $\operatorname{Hom}_F(V, V') \otimes_F \operatorname{Hom}_F(W, W')$ *to* $\operatorname{Hom}_F(V \otimes_F W, V' \otimes_F W')$. *Is it onto?*

Exercise ([1] §11.2 Proposition 17)**.** *Let* $A = (a_{ij})$ *be an* $m \times m'$ *matrix and* $B = (b_{ij})$ *be an* $n \times n'$ *matrix. The* ***tensor product*** *(or the* ***Kronecker product****) of two matrices is defined as an* $mn \times m'n'$ *matrix by*

$$A \otimes B := (a_{ij}B) = \begin{pmatrix} a_{11}B & \cdots & a_{1m'}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mm'}B \end{pmatrix}$$

consisting of an $m \times m'$ *block matrix whose* i, j *block is the* $n \times n'$ *matrix* $a_{ij}B$. *Explain why this definition is reasonable.*

Exercise. *If* V *and* W *are finite-dimensional vector spaces over* F*, show that*

$$\sum_{i=1}^{r} f_i \otimes w_i \mapsto \left(v \mapsto \sum_{i=1}^{r} f_i(v) w_i \right)$$

defines a canonical isomorphism $V^* \otimes_F W \cong \operatorname{Hom}_F(V, W)$. *What is the inverse of this map?*

2 Tensor algebras

From now on, all vector spaces are defined over a fixed field F and we drop all the subscripts of the tensor products.

2.1 Algebras over a field

The definition of an R-algebra where R is an arbitrary commutative ring with identity can be found in [1] §10.1, and will be covered in future lectures.

If F is a field, an **algebra** over F is a vector space V over F with a multiplication or product operation $V \times V \to V$ that is F-bilinear. The additive part of the F-bilinearity means that the product operation satisfies the distributive laws

$$a(b+c) = ab + ac \quad \text{and} \quad (b+c)a = ba + ca \quad \text{for all } a, b, c \in V,$$

and the scalar-multiplication part of the F-bilinearity means that

$$(\lambda a)b = \lambda(ab) = a(\lambda b) \quad \text{for all } \lambda \in F \text{ and } a, b \in V.$$

An algebra is determined by its vector-space structure and the multiplication table for the members of an F-basis.

In this course we shall work mostly just with **associative algebras**, i.e., those algebras satisfying the usual associative law

$$a(bc) = (ab)c \quad \text{for all } a, b, c \in V.$$

An associative algebra is therefore a ring and a vector space, the scalar multiplication and the ring multiplication being linked by the requirement that $(\lambda a)b = \lambda(ab) = a(\lambda b)$ for all scalars λ.

Example (commutative)**.** *The field extension K/F; the polynomial algebra $F[x_1, \ldots, x_n]$.*

Example (noncommutative)**.** *The matrix algebra $M_n(F)$; $\mathrm{End}_F(V) := \mathrm{Hom}_F(V, V)$ for any vector space V; the quaternion algebra $\mathbb{H}$; the group algebra $F[G]$.*

Exercise ([1] §10.4 Proposition 21)**.** *How to define an algebra structure on the tensor product of two algebras over F?*

2.2 Properties of tensor products

The next result shows that we may write $V_1 \otimes V_2 \otimes V_3$, or more generally, an n-fold tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_n$, unambiguously whenever it is defined.

Theorem ([1] §10.4 Theorem 14, Associativity of the Tensor Product)**.** *Suppose V_1, V_2, V_3 are vector spaces over the field F. Then there is a unique isomorphism*

$$(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$$

of vector spaces such that $(v_1 \otimes v_2) \otimes v_3 \mapsto v_1 \otimes (v_2 \otimes v_3)$.

Proof. For each fixed $v_3 \in V_3$, the mapping $(v_1, v_2) \mapsto v_1 \otimes (v_2 \otimes v_3)$ is bilinear, so by the universal property there is a homomorphism $V_1 \otimes V_2 \to V_1 \otimes (V_2 \otimes V_3)$ with $v_1 \otimes v_2 \mapsto v_1 \otimes (v_2 \otimes v_3)$. This shows that the map from $(V_1 \otimes V_2) \otimes V_3$ to $V_1 \otimes (V_2 \otimes V_3)$ given by $(v_1 \otimes v_2, v_3) \mapsto v_1 \otimes (v_2 \otimes v_3)$ is well defined. Since it is easily seen to be bilinear, another application of the universal property implies that it induces a homomorphism $(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$ such that $(v_1 \otimes v_2) \otimes v_3 \mapsto v_1 \otimes (v_2 \otimes v_3)$. In a similar way we can construct a homomorphism in the opposite direction that is inverse to this one. This proves the isomorphism. □

The next theorem shows that tensor products commute with direct sums.

Recall. *What is the definition of direct sum $V \oplus W$ of two vector spaces? What is its dimension? Determine a basis of it.*

Theorem ([1] §10.4 Theorem 17, Tensor Products of Direct Sums)**.** *Suppose V, V', W, W' are vector spaces over the field F. Then there are unique isomorphisms*

$$(V \oplus V') \otimes W \cong (V \otimes W) \oplus (V' \otimes W)$$
$$V \otimes (W \oplus W') \cong (V \otimes W) \oplus (V \otimes W')$$

of vector spaces such that $(v, v') \otimes w \mapsto (v \otimes w, v' \otimes w)$ and $v \otimes (w, w') \mapsto (v \otimes w, v \otimes w')$ respectively.

Theorem ([1] §10.4 Proposition 20)**.** *Suppose V, V' are vector spaces over the field F. Then there is a unique isomorphisms*

$$V \otimes V' \cong V' \otimes V$$

of vector spaces such that $v \otimes v' \mapsto v' \otimes v$.

Remark. *When $V = V'$ it is not in general true that $v \otimes v' = v' \otimes v$ for $v, v' \in V$. We shall study "symmetric tensors" in the next section.*

2.3 Tensor algebras

Suppose V is a vector space over the field F and $v_1, v_2 \in V$. We have formally defined a "product" $v_1 v_2$ of elements of V by tensor product, and we have constructed a new vector space $V \otimes V$ generated by such "products" $v_1 \otimes v_2$. The "value" of this product is not in V, so this does not give an algebra structure on V itself. If, however, we iterate this by taking the "products" $v_1 v_2 v_3$ and $v_1 v_2 v_3 v_4$, and all finite sums of such products, we can construct an algebra containing V that is "universal" with respect to algebras containing V. (If V is an F-algebra and already have multiplication structure, we need to "forget" this multiplication to define this universal object.)

For each integer $k \geq 1$, define $\mathcal{T}^k(V)$ (or $V^{\otimes k}$) by

$$\mathcal{T}^k(V) = \underbrace{V \otimes V \otimes \cdots \otimes V}_{k \text{ factors}},$$

and set $\mathcal{T}^0(V) = F$. The elements of $\mathcal{T}^k(V)$ are called **k-tensors**. Define

$$\mathcal{T}(V) = F \oplus \mathcal{T}^1(V) \oplus \mathcal{T}^2(V) \oplus \cdots = \bigoplus_{k=0}^{\infty} \mathcal{T}^k(V).$$

Every element of $\mathcal{T}(V)$ is a finite linear combination of k-tensors for various $k \geq 0$. We identify V with $\mathcal{T}^1(V)$, so that V is a subspace of $\mathcal{T}(V)$.

Theorem ([1] §11.5 Theorem 31)**.** *Let V be a vector space over the field F.*

(1) $\mathcal{T}(V)$ is an (associative) F-algebra containing V with multiplication defined by mapping

$$(v_1 \otimes \cdots \otimes v_i) \cdot (v_1' \otimes \cdots \otimes v_j') := v_1 \otimes \cdots \otimes v_i \otimes v_1' \otimes \cdots \otimes v_j'$$

*and extended to sums via the distributive laws. The algebra $\mathcal{T}(V)$ is called the **tensor algebra** of V. With respect to this multiplication $\mathcal{T}^i(V)\mathcal{T}^j(V) \subseteq \mathcal{T}^{i+j}(V)$.*

(2) (Universal Property) If A is any F-algebra and $\varphi : V \to A$ is an F-linear transformation, then there is a unique F-algebra homomorphism $\tilde{\varphi} : \mathcal{T}(V) \to A$ such that $\tilde{\varphi}|_V = \varphi$ and the diagram

$$\begin{array}{ccccc} \textit{(vector space)} & V & \hookrightarrow & \mathcal{T}(V) & \textit{(algebra)} \\ & & \searrow^{\varphi} & \downarrow \tilde{\varphi} & \\ & & & A & \textit{(algebra)} \end{array}$$

commutes.

Proposition ([1] §11.5 Proposition 32)**.** *Let V be a finite-dimensional vector space over the field F with basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. Then the k-tensors*

$$e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \quad \textit{with } e_{i_j} \in \mathcal{B}$$

are a vector space basis of $\mathcal{T}^k(V)$ over F (with the understanding that the basis vector is the element $1 \in F$ when $k = 0$). In particular, $\dim_F(\mathcal{T}^k(V)) = n^k$.

The above theorem and proposition show that the space $\mathcal{T}(V)$ may be regarded as the noncommutative polynomial algebra over F in the (noncommuting) variables $e_1, \ldots, e_n$.

Example. *When $V = F$ has dimension 1, $\mathcal{T}(F)$ is isomorphic to the polynomial algebra $F[x]$.*

Since $\mathcal{T}^i(V)\mathcal{T}^j(V) \subseteq \mathcal{T}^{i+j}(V)$, the tensor algebra $\mathcal{T}(V)$ has a natural "grading" or "degree" structure reminiscent of a polynomial ring. A ring S is called a **graded ring** if it is the direct sum of additive subgroups $S = S_0 \oplus S_1 \oplus S_2 \oplus \cdots$ such that $S_iS_j \subseteq S_{i+j}$ for all $i, j \geq 0$. The elements of S_k are said to be **homogeneous** of degree k, and S_k is called the **homogeneous component** of S of degree k.

Example. *The polynomial ring $S = R[x_1, x_2, \ldots, x_n]$ in n variables over the commutative ring R is a graded ring. Here $S_0 = R$ is a subring of S, and the homogeneous component of degree k is the additive subgroup of all R-linear combinations of monomials of degree k.*

3 Symmetric and exterior algebras

3.1 Symmetric algebras

Suppose V is a vector space over the field F. We have construct an algebra containing V that is "universal" with respect to algebras containing V, the tensor algebra. But it is in general noncommutative. How can we construct a "universal" commutative algebra containing V? The idea comes from forming the commutator quotient G/G' of a group.

The **symmetric algebra** of V is the F-algebra obtained by taking the quotient of the tensor algebra $\mathcal{T}(V)$ by the ideal $\mathcal{C}(V)$ generated by all elements of the form $v_1 \otimes v_2 - v_2 \otimes v_1$, for all $v_1, v_2 \in V$. The symmetric algebra $\mathcal{T}(V)/\mathcal{C}(V)$ is denoted by $\mathcal{S}(V)$.

Example. *(1) When $V = F$ has dimension 1, all the commutators are $0 \in F \otimes F$. The ideal $\mathcal{C}(F) = 0$ and the symmetric algebra $\mathcal{S}(F) = \mathcal{T}(F) \cong F[x]$.*

(2) When $V = Fe_1 \oplus Fe_2$ has dimension 2, all the commutators are constant multiples of $e_1 \otimes e_2 - e_2 \otimes e_1$. So in the quotient $\mathcal{S}(V) = \mathcal{T}(V)/\mathcal{C}(V)$, the coset of $e_1 \otimes e_2$ is the same with that of $e_2 \otimes e_1$, that means, in $\mathcal{S}(V)$ we can permute the order of e_1 and e_2.

Recall that $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ with $i_j \in \{1, 2\}$ are a vector space basis of $\mathcal{T}^k(V)$. Therefore (the cosets of) the same pure tensors also form a basis of $\mathcal{S}^k(V)$, with the order of the factors permuted are identified, i.e., (the cosets of) the pure tensors $e_1^{\otimes k_1} \otimes e_2^{\otimes k_2}$ with $k_1 + k_2 = k$ are a basis of $\mathcal{S}^k(V)$. Furthurmore $\mathcal{S}(V) \cong F[x_1, x_2]$.

Exercise. *Show that the symmetric algebra $\mathcal{S}(V) = \mathcal{T}(V)/\mathcal{C}(V)$ is graded with $\mathcal{S}^0(V) = F$ and $\mathcal{S}^1(V) = V$. We call the subspace $\mathcal{S}^k(V)$ the k^{th}* ***symmetric power*** *of V.*

Theorem ([1] §11.5 Theorem 34)**.** *Let V be a vector space over the field F and let $\mathcal{S}(V)$ be its symmetric algebra.*

(1) The k^{th} symmetric power $\mathcal{S}^k(V)$ of V is equal to $M^{\otimes k}$ modulo the submodule generated by all elements of the form

$$(v_1 \otimes \cdots \otimes v_k) - (v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)})$$

for all $v_i \in V$ and all permutations σ in the symmetric group S_k.

(2) (Universal property for maps to commutative algebras) If A is any commutative F-algebra and $\varphi : V \to A$ is a linear map, then there is a unique F-algebra homomorphism $\tilde{\varphi} : \mathcal{S}(V) \to A$ such that $\tilde{\varphi}|_V = \varphi$.

Corollary ([1] §11.5 Corollary 35)**.** *Let V be an n-dimensional vector space over the field F. Then $\mathcal{S}(V)$ is isomorphic as a graded F-algebra to the (commutative) ring of polynomials $F[x_1, \ldots, x_n]$ in n variables over F (i.e. the isomorphism is also a vector space isomorphism from $\mathcal{S}^k(V)$ onto the space of all homogeneous polynomials of degree k). In particular,* $\dim_F(\mathcal{S}^k(V)) = \binom{k+n-1}{n-1}$.

Example. *When $V = Fe_1 \oplus Fe_2$ is 2-dimensional, (the cosets of) $e_1 \otimes e_1$, $e_1 \otimes e_2$, $e_2 \otimes e_2$ form a basis of the 3-dimensional subspace $\mathcal{S}^2(V)$ of $\mathcal{S}(V)$.*

3.2 Exterior algebras

The **exterior algebra** of a vector space V over F, also called an **alternating algebra** or **Grassmann algebra**, is the F-algebra obtained by taking the quotient of the tensor algebra $\mathcal{T}(V)$ by the ideal $\mathcal{A}(V)$ generated by all elements of the form $v_1 \otimes v_2 + v_2 \otimes v_1$, for $v_1, v_2 \in V$. The exterior algebra $\mathcal{T}(V)/\mathcal{A}(V)$ is denoted by $\bigwedge(V)$ and the image of $v_1 \otimes \cdots \otimes v_k$ in $\bigwedge(V)$ is denoted by mt $v_1 \wedge \cdots \wedge v_k$.

Exercise. *Show that $\mathcal{A}(V)$ is equal to the ideal generated by all elements of the form $v \otimes v$ for $v \in V$.*

As with the symmetric algebra, the exterior algebra is also graded, with k^{th} homogeneous component $\bigwedge^k(V) = \mathcal{T}^k(V)/\mathcal{A}^k(V)$. We can again identify F with $\bigwedge^0(V)$ and V with $\bigwedge^1(V)$ and so consider V as a subspace of the F-algebra $\bigwedge(V)$. The subspace $\bigwedge^k(V)$ is called the k^{th} **exterior power** of V.

The multiplication

$$(v_1 \wedge \cdots \wedge v_i) \wedge (v_1' \wedge \cdots \wedge v_j') = v_1 \wedge \cdots \wedge v_i \wedge v_1' \wedge \cdots \wedge v_j'$$

in the exterior algebra is called the **wedge** (or **exterior**) **product**. By definition of the quotient, this multiplication is anticommutative on simple tensors:

$$v \wedge v' = -v' \wedge v \quad \text{for all } v, v' \in V.$$

The multiplication is also alternating in the sense that the product $v_1 \wedge \cdots \wedge v_k$ is 0 in $\bigwedge(V)$ if $v_i = v_j$ for some $i \neq j$ (cf. [1] §11.5 Theorem 36).

Exercise ([1] §11.5 Exercise 4). *The anticommutativity of wedge product does not extend to arbitrary products. It even does not extend to products of all simple tensors. Show that* $v_1 \wedge (v_2 \wedge v_3) = (v_2 \wedge v_3) \wedge v_1$ *for* $v_1, v_2, v_3 \in V$.

Example. *(1) Suppose V is a one-dimensional vector space over F with basis element e. Then $e \wedge e = 0$ implies that, the image $\lambda_1 e \wedge \cdots \wedge \lambda_k e$, of any simple tensors $\lambda_1 e \otimes \cdots \otimes \lambda_k e \in \mathcal{T}^k(V)$, is zero in $\bigwedge^k(V)$ for $k \geq 2$.*

It follows that $\bigwedge^0(V) = F$, $\bigwedge^1(V) = V$, and $\bigwedge^k(V) = 0$ for $k \geq 2$. As a graded F-algebra we have

$$\bigwedge(V) = F \oplus V \oplus 0 \oplus 0 \oplus \cdots .$$

(2) Suppose now that V is a two-dimensional vector space over F with basis e, e'.

Then $\bigwedge^k(V)$ consists of finite sums of elements of the form $v_1 \wedge \cdots \wedge v_k = (\lambda_1 e + \mu_1 e') \wedge \cdots \wedge (\lambda_k e + \mu_k e')$. Such an element is a sum of elements that are simple wedge products involving only e and e'. For example, an element in $\bigwedge^2(V)$ is a sum of elements of the form

$$\begin{aligned}(\lambda_1 e + \mu_1 e') \wedge (\lambda_2 e + \mu_2 e') &= \lambda_1\lambda_2(e \wedge e) + \lambda_1\mu_2(e \wedge e') + \mu_1\lambda_2(e' \wedge e) + \mu_1\mu_2(e' \wedge e') \\ &= (\lambda_1\mu_2 - \mu_1\lambda_2)(e \wedge e').\end{aligned}$$

It follows that $\bigwedge^k(V)$ for $k \geq 3$ since then at least one of e, e' appears twice in such simple products.

We have seen that the tensors in the 2^{nd} exterior power of V are all constant multiples of $e \wedge e'$. Actually one can see that $e \wedge e'$ by comparing the dimensions of $\mathcal{T}^2(V)$ and $\mathcal{A}^2(V)$.

It follows that $\bigwedge^0(V) = F$, $\bigwedge^1(V) = V$, $\bigwedge^2(V) = F(e \wedge e')$, and $\bigwedge^k(V) = 0$ for $k \geq 3$. As a graded F-algebra we have

$$\bigwedge(V) = F \oplus (Fe \oplus Fe') \oplus F(e \wedge e') \oplus 0 \oplus 0 \oplus \cdots .$$

Exercise. *Suppose that V is a 2-dimensional vector space over F with basis e, e'. Determine bases of $\mathcal{C}^2(V) := \mathcal{C}(V) \cap \mathcal{T}^2(V)$ and $\mathcal{A}^2(V) := \mathcal{A}(V) \cap \mathcal{T}^2(V)$. Recall that $\mathcal{C}(V)$ and $\mathcal{A}(V)$ are the ideals in the definition of the symmetric and the exterior algebra respectively.*

As the previous examples illustrate, unlike the tensor and symmetric algebras, for finite-dimensional vector spaces the exterior algebra is finite dimensional.

Corollary ([1] §11.5 Corollary 37). *Let V be a finite-dimensional vector space over the field F with basis $\mathcal{B} = \{e_1, \ldots, e_n\}$. Then the vectors*

$$v_{i_1} \wedge \cdots \wedge v_{i_k} \quad \textit{for } 1 \leq i_1 < \cdots < i_k \leq n$$

are a basis of $\bigwedge^k(V)$, and $\bigwedge^k(V) = 0$ when $k > n$ (when $k = 0$ the basis vector is the element $1 \in F$). In particular, $\dim_F(\bigwedge^k(V)) = \binom{n}{k}$ and $\dim_F(\bigwedge(V)) = 2^n$.

Exercise. *How do you understand the wedge product of two 3-dimensional vectors? For $V = \mathbb{R}^3$, show that $v_1 \wedge v_2 \mapsto (v_3 \mapsto \det[v_1\ v_2\ v_3])$ defines an isomorphism $\bigwedge^2(V) \cong V^*$ of real vector spaces. Can you generalize this result to m-dimensional vector spaces for any m?*

3.3 Symmetric and alternating tensors

The symmetric and exterior algebras can in some instances also be defined in terms of symmetric and alternating tensors, which identify these algebras as subalgebras of the tensor algebra rather than as quotient algebras.

For any vector space V there is a natural left group action of the symmetric group S_k on V^k given by permuting the factors:

$$\sigma(v_1, \dots, v_k) = (v_{\sigma^{-1}(1)}, \dots v_{\sigma^{-1}(k)}) \quad \text{for each } \sigma \in S_k$$

(the reason for σ^{-1} is to make this a *left* group action). This map is clearly F-multilinear, so there is a well defined F-linear left group action of S_k on $\mathcal{T}^k(V)$ which is defined on simple tensors by

$$\sigma(v_1 \otimes \dots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(k)} \quad \text{for each } \sigma \in S_k.$$

An element $z \in \mathcal{T}^k(V)$ is called a **symmetric** k-tensor if $\sigma z = z$ for all σ in the symmetric group S_k; is called an **alternating** k-tensor if $\sigma z = \text{sgn}(\sigma) z$ for all $\sigma \in S_k$, where $\text{sgn}(\sigma)$ is the sign ± 1 of the permutation σ.

Example. *The elements $v \otimes v$ and $v_1 \otimes v_2 + v_2 \otimes v_1$ arc symmetric 2-tensors. The element $v_1 \otimes v_2 - v_2 \otimes v_1$ is an alternating 2-tensor.*

It is immediate from the definition that the collection of symmetric (respectively, alternating) k-tensors is a subspace of $\mathcal{T}^k(V)$.

Define linear transformations on $\mathcal{T}^k(V)$ by

$$\text{Sym}(z) := \sum_{\sigma \in S_k} \sigma(z), \quad \text{Alt}(z) := \sum_{\sigma \in S_k} \text{sgn}(\sigma)\sigma(z).$$

For any k-tensor z, the k-tensor $\text{Sym}(z)$ is symmetric and the k-tensor $\text{Alt}(z)$ is alternating. For example, for any $\tau \in S_k$,

$$\begin{aligned} \tau \,\text{Alt}(z) &= \sum_{\sigma \in S_k} \text{sgn}(\sigma)\tau\sigma(z) = \sum_{\sigma' \in S_k} \text{sgn}(\tau^{-1}\sigma')\sigma'(z) \quad (\text{letting } \sigma' = \tau\sigma) \\ &= \text{sgn}(\tau^{-1}) \sum_{\sigma' \in S_k} \text{sgn}(\sigma')\sigma'(z) = \text{sgn}(\tau)\,\text{Alt}(z). \end{aligned}$$

The tensor $\text{Sym}(z)$ is called the **symmetrization** of z and $\text{Alt}(z)$ the **skew-symmetrization** of z.

Proposition ([1] §11.5 Proposition 40)**.** *Suppose* $\text{char}(F) \nmid k!$ *and V is a vector space over F. Then*

(1) The map $\frac{1}{k!}$ Sym induces an isomorphism of vector spaces between the k^{th} symmetric power of V and the subspace of symmetric k-tensors:

$$\frac{1}{k!}\,\text{Sym} : \mathcal{S}^k(V) \xrightarrow{\sim} \{\textit{symmetric } k\textit{-tensors}\}.$$

(2) The map $\frac{1}{k!}$ Alt induces an isomorphism of vector spaces between the k^{th} exterior power of V and the subspace of alternating k-tensors:

$$\frac{1}{k!}\,\text{Alt} : \Lambda^k(V) \xrightarrow{\sim} \{\textit{alternating } k\textit{-tensors}\}.$$

Exercise. *Let V be a 3-dimensional vector space over a field F in which $6 \neq 0$. Suppose e_1, e_2, e_3 form a basis of V. Calculate $\frac{1}{6}\operatorname{Sym}(z \otimes w)$ and $\frac{1}{6}\operatorname{Alt}(z \otimes w)$ for $z = e_1$, $w = e_2 \otimes e_3 - e_3 \otimes e_2$.*

Exercise. *Suppose $\operatorname{char}(F) \nmid k!$ and V is a vector space over F. Show that $\pi^2 = \pi$ for both $\pi = \frac{1}{k!}\operatorname{Sym}$ and $\pi = \frac{1}{k!}\operatorname{Alt}$.*

When $k!$ is invertible, this exercise shows that π is a projection onto the subspace $\operatorname{image}(\pi)$ (cf. [1] §11.2 Exercise 11), and then we have direct sums of vector spaces

$$\mathcal{T}^k(V) = \ker(\pi) \oplus \operatorname{image}(\pi)$$

for $\pi = \frac{1}{k!}\operatorname{Sym}$ or $\pi = \frac{1}{k!}\operatorname{Alt}$. It is possible to define the k^{th} exterior power of V as the collection of alternating k-tensors. In this case the multiplication of two alternating tensors z and w is defined by first taking the product $zw := z \otimes w$ in $\mathcal{T}(V)$ and then projecting the resulting tensor into the subspace of alternating tensors. Note that the simple product of two alternating tensors need not be alternating (for example, the square of an alternating tensor is a symmetric tensor).

If $k!$ is not invertible then in general we do not have such S_k-invariant direct sum decompositions, so it is not in general possible to identify, for example, the k^{th} exterior power of V with the alternating k-tensors of V.

Other related exercises in [1]

§10.4 11, 12, 13, 27, 23 (assume that R is a field), 25 (assume that both R and S are fields)
§11.2 38, 39
§11.5 4, 10, 11, 12, 13

References

[1] D. S. Dummit and R. M. Foote. *Abstract algebra*. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004.

[2] A. W. Knapp. *Basic algebra*. Cornerstones. Birkhäuser Boston, Inc., Boston, MA, 2006. Along with a companion volume *Advanced algebra*.

[3] 聂灵沼，丁石孙. 代数学引论. 高等教育出版社，第三版，2021.