Shandong University Advanced Modern Algebra

Lecture 6: Introduction to Homological Algebra 1

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This lecture refers to §10.5 in [1]. All the equation numbers without reference labels are from this book.

1 Exact sequences

1.1 Exact sequences and diagram chasing

We first introduce a very convenient notation.

Definition. *Let X, Y, Z, X_i be some algebraic objects (e.g., groups, rings, or modules).*

*(1) The pair of homomorphisms $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ is said to be **exact** (at Y) if $\mathrm{image}\,\alpha = \ker\beta$.*

*(2) A **sequence** $\cdots \to X_{n-1} \to X_n \to X_{n+1} \to \cdots$ of homomorphisms is said to be an **exact sequence** if it is exact at every X_n between a pair of homomorphisms.*

*(3) The exact sequence $0 \to X \xrightarrow{\psi} Y \xrightarrow{\varphi} Z \to 0$ is called a **short exact sequence**. (If X, Y and Z are groups written multiplicatively, the sequence will be written $1 \to X \xrightarrow{\psi} Y \xrightarrow{\varphi} Z \to 1$ where 1 denotes the trivial group.) Y is called an **extension** of Z by X.*

Proposition ([1] §10.5 Proposition 22 & Corollary 23)**.** *Let X, Y, Z be some algebraic objects.*

(1) The sequence $0 \to X \xrightarrow{\psi} Y$ is exact (at X) if and only if ψ is injective (denoted by $X \overset{\psi}{\hookrightarrow} Y$).

(2) The sequence $Y \xrightarrow{\varphi} Z \to 0$ is exact (at Z) if and only if φ is surjective (denoted by $Y \overset{\varphi}{\twoheadrightarrow} Z$).

(3) The sequence $0 \to X \xrightarrow{\psi} Y \xrightarrow{\varphi} Z \to 0$ is exact if and only if

- *ψ is injective,* • *φ is surjective, and* • $\mathrm{image}\,\psi = \ker\varphi$.

Proof. The (uniquely defined) homomorphism $0 \to A$ has image 0 in A. This will be the kernel of ψ if and only if ψ is injective.

Similarly, the kernel of the (uniquely defined) zero homomorphism $C \to 0$ is all of C, which is the image of φ if and only if φ is surjective. □

Note that any exact sequence can be written as a succession of short exact sequences, since to say $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ is exact at Y is the same as saying that the sequence

$$0 \longrightarrow \alpha(X) \longrightarrow Y \longrightarrow Y/\ker\beta \longrightarrow 0$$

is a short exact sequence.

Example. *Given (left) R-modules A and C we can always form their direct sum $B = A \oplus C$, and the sequence*

$$0 \longrightarrow A \xrightarrow{\iota} A \oplus C \xrightarrow{\pi} C \longrightarrow 0$$

where $\iota(a) \coloneqq (a, 0)$ and $\pi(a, c) \coloneqq c$ is a short exact sequence.

This is also valid for groups (not necessarily abelian), for example,

$$1 \longrightarrow \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}^\times \longrightarrow \mathbb{C}^\times \longrightarrow 1$$

is a short exact sequence.

Example. *If $\varphi : B \to C$ is any homomorphism we may form an exact sequence:*

$$0 \longrightarrow \ker\varphi \xrightarrow{\iota} B \xrightarrow{\varphi} \varphi(B) \longrightarrow 0$$

where ι is the inclusion map. In particular, if φ is surjective, the sequence $B \xrightarrow{\varphi} C$ may be extended to a short exact sequence with $A = \ker\varphi$. For example,

$$1 \longrightarrow \mathrm{SL}_2(\mathbb{C}) \hookrightarrow \mathrm{GL}_2(\mathbb{C}) \xrightarrow{\det} \mathbb{C}^\times \longrightarrow 1$$

is a short exact sequence.

One particularly important instance is when M is an R-module and S is a set of generators for M. Let $F(S)$ be the free R-module on S. Then

$$0 \longrightarrow K \xrightarrow{\iota} F(S) \xrightarrow{\varphi} M \longrightarrow 0$$

is the short exact sequence where φ is the unique R-module homomorphism which is the identity on S (cf. [1] §10.3 Theorem 6, the universal property) and $K = \ker\varphi$.

More generally, when M is any group (possibly nonabelian), the above short exact sequence (with 1's at the ends, if M is written multiplicatively) describes a presentation of M, where K is the normal subgroup of $F(S)$ generated by the relations defining M (cf. [1] §6.3). For example,

$$1 \longrightarrow H \longrightarrow \langle \sigma, \tau \rangle \longrightarrow D_{2n} = \langle \sigma, \tau \mid \sigma^n = \tau^2 = e, \tau\sigma\tau = \sigma^{-1} \rangle \longrightarrow 1$$

is the short exact sequence where $\langle \sigma, \tau \rangle$ is the free group on $\{\sigma, \tau\}$, and H is the smallest normal subgroup of $\langle \sigma, \tau \rangle$ containing σ^n, τ^2, and $\sigma\tau\sigma\tau$.

Let $0 \to A \to B \to C \to 0$ and $0 \to A' \to B' \to C' \to 0$ be two short exact sequences of modules. A **homomorphism of short exact sequences** is a triple α, β, γ of module homomorphisms such that the following diagram commutes:

$$\begin{CD}
0 @>>> A @>>> B @>>> C @>>> 0 \\
@. @V{\alpha}VV @V{\beta}VV @V{\gamma}VV @. \\
0 @>>> A' @>>> B' @>>> C' @>>> 0
\end{CD}$$

The homomorphism is an **isomorphism of short exact sequences** if α, β, γ are all isomorphisms, in which case the extensions B and B' are said to be **isomorphic extensions**. The two exact sequences are called **equivalent** if $A = A'$, $C = C'$, and there is an isomorphism between them that is the identity maps on A and C (i.e., α and γ are the identity). This means the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \Vert & & \beta \downarrow \cong & & \Vert & & \\ 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

i.e.,

$$\begin{array}{ccccccc} & & & B & & & \\ & & \nearrow & & \searrow & & \\ 0 \longrightarrow & A & & \beta \downarrow \cong & & C & \longrightarrow 0 \\ & & \searrow & & \nearrow & & \\ & & & B' & & & \end{array}$$

In this case the corresponding extensions B and B' are said to be **equivalent extensions**.

Proposition ([1] §10.5 Proposition 24, The Short Five Lemma)**.** *Let α, β, γ be a homomorphism of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\psi} & B & \xrightarrow{\varphi} & C & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & A' & \xrightarrow{\psi'} & B' & \xrightarrow{\varphi'} & C' & \longrightarrow & 0 \end{array}$$

(1) If α and γ are injective then so is β.

(2) If α and γ are surjective then so is β.

(3) If α and γ are isomorphisms then so is β (and then the two sequences are isomorphic).

These results hold also for short exact sequences of (possibly nonabelian) groups.

Proof. (1) Suppose then that α and γ are injective and suppose $b \in B$ with $\beta(b) = 0$. It follows in particular that the image of $\beta(b)$ in C' is also 0.

By the commutativity of the diagram this implies that $\gamma(\varphi(b)) = 0$, and since γ is assumed injective, we obtain $\varphi(b) = 0$, i.e., b is in the kernel of φ.

By the exactness of the first sequence, this means that b is in the image of ψ, i.e., $b = \psi(a)$ for some $a \in A$.

Then, again by the commutativity of the diagram, the image of $\alpha(a)$ in B' is the same as $\beta(\psi(a)) = \beta(b) = 0$.

But α and ψ' are injective by assumption, it follows that $a = 0$.

Finally, $b = \psi(a) = \psi(0) = 0$ and we see that β is indeed injective.

(2) Exercise.

(3) follows immediately from (1) and (2). □

The proving technique above is called **diagram chasing**, which consists in looking for equivalent map compositions in commutative diagrams, and in exploiting the properties of injective, surjective and bijective homomorphisms and of exact sequences.

Exercise (3×3 Lemma)**.** *Suppose*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

is a commutative diagram of R-modules with exact columns. Show that:

(a) if the bottom two rows are exact then so is the top row;

(b) if the top two rows are exact then so is the bottom row;

(c) if the top and bottom rows are exact, and the composite $A \to C$ is zero, then the middle row is also exact.

Exercise ([1] §17.1 Exercise 3)**.** *Suppose*

$$\begin{array}{ccccccc}
 & & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\
 & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \\
0 & \longrightarrow & A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & &
\end{array}$$

is a commutative diagram of R-modules with exact rows.

(a) If $c \in \ker(h)$ and $\beta(b) = c$, prove that $g(b) \in \ker \beta'$ and conclude that $g(b) = \alpha'(a')$ for some $a' \in A'$.

(b) Show that $\delta(c) := a' \bmod f(A)$ is a well defined R-module homomorphism from $\ker(h)$ to $\operatorname{coker}(f) := A'/f(A)$ (the ***cokernel*** *of f).*

(c) Show that

$$\ker(f) \xrightarrow{\alpha|_{\ker f}} \ker(g) \xrightarrow{\beta|_{\ker g}} \ker(h) \quad \textit{and} \quad \operatorname{coker}(f) \xrightarrow{\bar{\alpha}} \operatorname{coker}(g) \xrightarrow{\bar{\beta}} \operatorname{coker}(h)$$

are well defined and exact.

(d) (The Snake Lemma) *Prove there is an exact sequence*

$$\ker(f) \longrightarrow \ker(g) \longrightarrow \ker(h) \xrightarrow{\delta} \operatorname{coker}(f) \longrightarrow \operatorname{coker}(g) \longrightarrow \operatorname{coker}(h).$$

(e) Show that if α is injective and β' is surjective (i.e., the two rows in the commutative diagram above can be extended to short exact sequences), then the exact sequence above can be extended to the exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow \ker(g) \longrightarrow \ker(h) \xrightarrow{\delta} \operatorname{coker}(f) \longrightarrow \operatorname{coker}(g) \longrightarrow \operatorname{coker}(h) \longrightarrow 0.$$

1.2 The extension problem

We have done an exercise ([1] §3.1 Exercise 36) that, for any group G, if $G/Z(G)$ is cyclic then G is abelian. The philosophy behind this is that, if we have a sufficient amount of information about some normal subgroup, N, of a group G and sufficient information on G/N, then somehow we can piece this information together to force G itself to have some desired property. In general, just how much data are required is a delicate matter since the full isomorphism type of G cannot be determined from the isomorphism types of N and G/N alone.

In the language of left R-modules where R is a ring with 1, we consider whether, given two modules A and C, there exists a module B containing (an isomorphic copy of) A such that the resulting quotient module B/A is isomorphic to C. In this case we have a short exact sequence $0 \to A \to B \to C \to 0$, and B is said to be an **extension of** C **by** A.

It is then natural to ask how many such B exist for a given A and C, and the extent to which properties of B are determined by the corresponding properties of A and C. There are, of course, analogous problems in the contexts of groups and rings.

Example. *There is always at least one extension of a module C by A, namely the direct sum $B = A \oplus C$. In particular,*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\varphi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

gives one extension of $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}$.

Another extension is given by the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

where $\times n$ denotes the map $x \mapsto nx$ given by multiplication by n, and π denotes the natural projection.

Note that the modules in the middle of the previous two exact sequences are not isomorphic even though the respective "A" and "C" terms are isomorphic. Thus there are (at least) two inequivalent ways of extending $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}$.

Recall that, if B and B' are **isomorphic extensions** of C by A, then in particular B and B' are isomorphic as R-modules, but more is true: there is an R-module isomorphism between B and B' that restricts to an isomorphism from A to A' and induces an isomorphism on the quotients C and C'. For a given A and C the condition that two extensions B and B' of C by A are **equivalent** is stronger still: there must exist an R-module isomorphism between B and B' that restricts to the identity map on A and induces the identity map on C.

The notion of isomorphic extensions measures how many different extensions of C by A there are, allowing for C and A to be changed by an isomorphism. The notion of equivalent extensions measures how many different extensions of C by A there are when A and C are rigidly fixed.

Example. *Consider the maps*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\psi} & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\varphi} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\psi'} & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\varphi'} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0
\end{array}$$

where ψ maps $\mathbb{Z}/2\mathbb{Z}$ injectively into the first component of the direct sum and φ projects the direct sum onto its second component. Also ψ' embeds $\mathbb{Z}/2\mathbb{Z}$ into the second component of the direct sum and φ' projects the direct sum onto its first component.

If β maps the direct sum $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ to itself by interchanging the two factors, then this diagram is seen to commute, hence giving an equivalence of the two exact sequences that is not the identity isomorphism.

1.3 The semidirect product of groups and the split extensions

Recall (cf. [1] §5.1) that, the **direct product** $G_1 \times G_2 \times \cdots$ of groups $G_1, G_2, \ldots$ with operations $\star_1, \star_2, \ldots$, respectively, is the set of sequences $(g_1, g_2, \ldots)$ where $g_i \in G_i$ with operation defined componentwise:

$$(g_1, g_2, \ldots) \star (h_1, h_2, \ldots) := (g_1 \star_1 h_1, g_2 \star_2 h_2, \ldots).$$

If $G = G_1 \times G_2 \times \cdots \times G_n$, then $G_i \trianglelefteq G$ and $G/G_i \cong \underset{j\neq i}{\times} G_j$ (cf. [1] §5.1 Proposition 2).

The "semidirect product" of two groups is a generalization of the notion of the direct product, obtained by relaxing the requirement that both groups should be normal subgroups of the product. This construction will enable us to build a "larger" group from the groups H and K in such a way that G contains subgroups isomorphic to H and K, respectively, as in the case of direct products. In this case the subgroup H will be normal in G but the subgroup K will not necessarily be normal. Thus, for instance, we shall be able to construct nonabelian groups even if H and K are abelian.

Example. *Let $G = D_{2n} = \langle \sigma, \tau \mid \sigma^n = \tau^2 = e, \tau\sigma\tau = \sigma^{-1} \rangle$ be the dihedral group, $H = \langle \sigma \rangle \cong \mathbb{Z}/n\mathbb{Z}$, and $K = \langle \tau \rangle \cong \mathbb{Z}/2\mathbb{Z}$.*

Clearly $G = HK$, $H \trianglelefteq G$ (since it is of index 2) but K may not be normal. By definition of a normal subgroup, $\tau H \tau^{-1} = H$, i.e. τ (and K in general) acts on H by conjugation. Let φ be the associated permutation representation $\varphi : K \to \mathrm{Aut}(H)$ such that $\varphi(\tau)(\sigma) = \tau\sigma\tau^{-1} = \sigma^{-1}$ by the definition of D_{2n}.

Any element in D_{2n} can uniquely be written as $\sigma^a\tau^b$ for some $0 \leq a < n$, $b = 0$ or 1. The product of any two elements in D_{2n} can be calculated as following:

$$\sigma^{a_1}\tau^{b_1}\sigma^{a_2}\tau^{b_2} = \begin{cases} \sigma^{a_1}\sigma^{a_2}\tau^{b_1}\tau^{b_2} & \text{if } b_1 = 0, \\ \sigma^{a_1}(\sigma^{-1})^{a_2}\tau^{b_1}\tau^{b_2} & \text{if } b_1 = 1. \end{cases}$$

Example. *Let $G = \mathrm{GL}_2(\mathbb{C})$, $H = \mathrm{SL}_2(\mathbb{C})$, and*

$$K = \left\{ \begin{bmatrix} \lambda & \\ & 1 \end{bmatrix} : \lambda \in \mathbb{C}^\times \right\} \cong \mathbb{C}^\times.$$

Again we have $H \trianglelefteq G$ (since $H = \ker\det$) and $G = HK$ (since $g\left[\begin{smallmatrix} \det g & \\ & 1 \end{smallmatrix}\right]^{-1} \in H$ for any $g \in G$). Analogous to the previous example, K acts by conjugation on H and the associated permutation representation is denoted by $\varphi : K \to \mathrm{Aut}(H)$. Here

$$\varphi(\begin{bmatrix} \lambda & \\ & 1 \end{bmatrix}) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \lambda & \\ & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda & \\ & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a & b\lambda \\ c/\lambda & d \end{bmatrix}.$$

Since $H \cap K = 1$, every element of $G = HK$ can be written uniquely as a product hk for some $h \in H$ and $k \in K$, i.e., there is a bijection between G and the collection of ordered pairs (h, k), given by $hk \mapsto (h, k)$. The product of any two elements in G can also be written in the same form:

$$(h_1k_1)(h_2k_2) = h_1(k_1h_2)k_2 = h_1(k_1h_2k_1^{-1})k_1k_2 = h_1(\varphi(k_1)h_2)k_1k_2.$$

Theorem ([1] §5.5 Theorems 10 & 12)**.** *Let H and K be groups and let $\varphi : K \to \mathrm{Aut}(H)$ be a homomorphism. Let G be the set of ordered pairs (h, k) with $h \in H$ and $k \in K$ and define the following multiplication on G:*

$$(h_1, k_1)(h_2, k_2) := (h_1 \cdot \varphi(k_1)h_2, k_1k_2).$$

*This multiplication makes G into a group, which is called the **semidirect product** of H and K with respect to φ and will be denoted by $H \rtimes_\varphi K$ (when there is no danger of confusion we shall simply write $H \rtimes K$).*

The sets $\{(h, 1) \mid h \in H\}$ and $\{(1, k) \mid k \in K\}$ are subgroups of G and the maps $h \mapsto (h, 1)$ for $h \in H$ and $k \mapsto (1, k)$ for $k \in K$ are isomorphisms:

$$H \cong \{(h, 1) \mid h \in H\} \quad \textit{and} \quad K \cong \{(1, k) \mid k \in K\}.$$

Identifying H and K with their isomorphic copies in G, we have

$$H \trianglelefteq G, \quad H \cap K = 1, \quad G/H \cong K,$$

and $khk^{-1} = \varphi(k)h$ for all $h \in H$ and $k \in K$.

Conversely, suppose that G is a group with subgroups H and K such that $G = HK$, $H \trianglelefteq G$, and $H \cap K = 1$. Let $\varphi : K \to \mathrm{Aut}(H)$ be the homomorphism defined by mapping $k \in K$ to the automorphism of left conjugation by k on H. Then $G \cong H \rtimes K$.

For the semidirect product we can write a short exact sequence $1 \to H \to H \rtimes K \to K \to 1$.

It is easy to check that the semidirect product is a generalization of the direct product: the semidirect product $G = H \rtimes K$ is isomorphic to $H \times K$ if and only if φ is trivial, if and only if $K \trianglelefteq G$ (cf. [1] §5.5 Proposition 11).

Exercise ([1] §5.5 p.179 & p.181)**.** *Classify all groups of order pq up to isomorphism, where p and q are primes with $p < q$.*

We come back to the extension question. Recall that there is always at least one extension of a left R-module C by A, namely the direct sum $B = A \oplus C$. In this case the module B contains a submodule C' isomorphic to C (namely $C' = 0 \oplus C$) as well as the submodule A, and this submodule complement to A "splits" B into a direct sum. In the case of groups the existence of a subgroup complement C' to a normal subgroup A in B implies that B is a semidirect product $A \rtimes C$. The fact that B is a direct sum in the context of modules is a reflection of the fact that the underlying group structure in this case is abelian; for abelian groups semidirect products are direct products. In either case the corresponding short exact sequence is said to "split":

Proposition ([1] §10.5 Proposition 25)**.** *The short exact sequence $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ is called **split** if there is a homomorphism $\mu : C \to B$ such that $\varphi \circ \mu$ is the identity map on C. Such μ is called a **splitting homomorphism** for the sequence. The extension B is said to be a **split extension** of C by A.*

- *If $0 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 0$ is a short exact sequence of R-modules, then the sequence is split if and only if there is an R-module complement to $\psi(A)$ in B. In this case, up to isomorphism, $B = A \oplus C$ (more precisely, $B = \psi(A) \oplus C'$ for some submodule C', and C' is mapped isomorphically onto C by φ).*

- *If $1 \to A \xrightarrow{\psi} B \xrightarrow{\varphi} C \to 1$ is a short exact sequence of groups, then the sequence is split if and only if there is a subgroup complement to $\psi(A)$ in B. In this case, up to isomorphism, $B = A \rtimes C$ (more precisely, $B = \psi(A) \rtimes C'$ for some subgroup C', and C' is mapped isomorphically onto C by φ).*

Any set map $\mu : C \to B$ such that $\varphi \circ \mu = \mathrm{id}$ is called a **section** of φ. Note that a section of φ is nothing more than a choice of coset representatives in B for the quotient $B/\ker\varphi \cong C$. A section is a (splitting) homomorphism if this set of coset representatives forms a submodule (respectively, subgroup) in B, in which case this submodule (respectively, subgroup) gives a complement to $\psi(A)$ in B.

Example. *The extension*

$$0 \longrightarrow \mathbb{Z} \stackrel{\iota}{\longrightarrow} \mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z}) \stackrel{\varphi}{\longrightarrow} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

of $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}$ is split (with splitting homomorphism μ mapping $\mathbb{Z}/n\mathbb{Z}$ isomorphically onto the second factor of the direct sum).

On the other hand, the exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z} \stackrel{\times n}{\longrightarrow} \mathbb{Z} \stackrel{\pi}{\longrightarrow} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

is not split since there is no nonzero homomorphism of $\mathbb{Z}/n\mathbb{Z}$ into $\mathbb{Z}$.

The next proposition shows in particular that, for modules, the existence of a splitting homomorphism (for φ) is equivalent to the existence of a splitting homomorphism for ψ at the other end of the sequence.

Proposition ([1] §10.5 Proposition 26)**.** *If $0 \to A \stackrel{\psi}{\to} B \stackrel{\varphi}{\to} C \to 0$ is a short exact sequence of modules, then $B = \psi(A) \oplus C'$ for some submodule C' of B with $\varphi(C') \cong C$ if and only if there is a homomorphism $\lambda : B \to A$ such that $\lambda \circ \psi$ is the identity map on A.*

If $1 \to A \stackrel{\psi}{\to} B \stackrel{\varphi}{\to} C \to 1$ is a short exact sequence of groups, then $B = \psi(A) \times C'$ for some subgroup C' of B with $\varphi(C') \cong C$ if and only if there is a homomorphism $\lambda : B \to A$ such that $\lambda \circ \psi$ is the identity map on A.

The above proposition shows that, for general group extensions, the existence of a splitting homomorphism A on the left end of the sequence is stronger than the condition that the extension splits: in this case the extension group is a direct product, and not just a semidirect product. The fact that these two notions are equivalent in the context of modules is again a reflection of the abelian nature of the underlying groups, where semidirect products are always direct products.

2 The functors $\mathrm{Hom}_R(D, \_)$ and $\mathrm{Hom}_R(\_, D)$

Let R be a ring with 1 and suppose the R-module M is an extension of N by L, with $0 \to L \stackrel{\psi}{\to} M \stackrel{\varphi}{\to} N \to 0$ the corresponding short exact sequence of R-modules. It is natural to ask whether properties for L and N imply related properties for the extension M.

2.1 Projective modules and $\mathrm{Hom}_R(D, \_)$

The first situation we shall consider is whether an R-module homomorphism from some fixed R-module D to either L or N implies there is also an R-module homomorphism from D to M.

The question of obtaining a homomorphism from D to M given a homomorphism from D to L is easily disposed of: if $f \in \mathrm{Hom}_R(D, L)$ is an R-module homomorphism from D to L, then the composite $f' = \psi \circ f$ is an R-module homomorphism from D to M.

The relation between these maps can be indicated pictorially by the commutative diagram

$$\begin{array}{ccc} D & & \\ {\scriptstyle f}\downarrow & \searrow^{f'} & \\ L & \xrightarrow{\psi} & M \end{array}$$

Put another way, composition with ψ induces a map

$$\begin{aligned} \psi_* : \mathrm{Hom}_R(D, L) &\longrightarrow \mathrm{Hom}_R(D, M) \\ f &\longmapsto f' := \psi \circ f. \end{aligned}$$

Proposition ([1] §10.5 Proposition 27)**.** *Let D, L and M be R-modules and let $\psi : L \to M$ be an R-module homomorphism. Then the map $\psi_* : \mathrm{Hom}_R(D, L) \to \mathrm{Hom}_R(D, M)$ defined above is a homomorphism of abelian groups.*

If ψ is injective, then ψ_* is also injective, i.e.,

$$\textit{if } \; 0 \longrightarrow L \xrightarrow{\psi} M \; \textit{ is exact,}$$
$$\textit{then } \; 0 \longrightarrow \mathrm{Hom}_R(D, L) \xrightarrow{\psi_*} \mathrm{Hom}_R(D, M) \; \textit{ is also exact.}$$

Proof. The fact that ψ_* is a homomorphism is immediate.

If ψ is injective, then distinct homomorphisms f and g from D into L give distinct homomorphisms $\psi \circ f$ and $\psi \circ g$ from D into M, which is to say that ψ_* is also injective. □

If D is fixed, then given any R-module X we have an associated abelian group $\mathrm{Hom}_R(D, X)$. Further, an R-module homomorphism $\alpha : X \to Y$ induces an abelian group homomorphism $\alpha_* : \mathrm{Hom}_R(D, X) \to \mathrm{Hom}_R(D, Y)$, defined by $\alpha_*(f) := \alpha \circ f$. Put another way, the map $\mathrm{Hom}_R(D, \_)$ is a **covariant functor** from the **category** of R-modules to the category of abelian groups (cf. [1] Appendix II).

The situation for homomorphisms into the quotient is much less evident. More precisely, given an R-module homomorphism $f : D \to N$, the question is whether there exists an R-module homomorphism $F : D \to M$ that **extends** or **lifts** f to M, i.e., that makes the following diagram commute:

$$\begin{array}{ccc} & & D \\ & {}^{F}\swarrow & \downarrow{\scriptstyle f} \\ M & \xrightarrow{\varphi} & N \end{array}$$

As before, composition with the homomorphism φ induces a homomorphism of abelian groups

$$\begin{aligned} \varphi_* : \mathrm{Hom}_R(D, M) &\longrightarrow \mathrm{Hom}_R(D, N) \\ F &\longmapsto F' := \varphi \circ F. \end{aligned}$$

In terms of φ_*, the homomorphism f to N lifts to a homomorphism to M if and only if f is in the image of φ_* (namely, f is the image of the lift F).

In general it may not be possible to lift a homomorphism f from D to N to a homomorphism from D to M. Phrased in terms of the map φ_*, this shows that

$$\text{if } \; M \xrightarrow{\varphi} N \longrightarrow 0 \; \text{ is exact,}$$
$$\text{then } \; \mathrm{Hom}_R(D, M) \xrightarrow{\varphi_*} \mathrm{Hom}_R(D, N) \longrightarrow 0 \; \text{ is } \underline{\text{not necessarily}} \text{ exact.}$$

Example. *Consider the nonsplit exact sequence* $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0$. *Let* $D = \mathbb{Z}/2\mathbb{Z}$ *and let* f *be the identity map from* D *into* N. *Any homomorphism* F *of* D *into* $M = \mathbb{Z}$ *must map* D *to* 0 *(since* $\mathbb{Z}$ *has no elements of order* 2*), hence* $\pi \circ F$ *maps* D *to* 0 *in* N*, and in particular,* $\pi \circ F \neq f$.

These results relating the homomorphisms into L and N to the homomorphisms into M can be neatly summarized as part of the following theorem.

Theorem ([1] §10.5 Theorem 28 & Corollary 32)**.** *Let* D*,* L*,* M*, and* N *be* R*-modules.*

(1) If $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ *is exact, then the associated sequence*

$$0 \longrightarrow \operatorname{Hom}_R(D, L) \xrightarrow{\psi_*} \operatorname{Hom}_R(D, M) \xrightarrow{\varphi_*} \operatorname{Hom}_R(D, N) \qquad \textit{is exact}, \tag{10.10}$$

i.e., the functor $\operatorname{Hom}_R(D, \_)$ *from the category of* R*-modules to the category of abelian groups is* ***left exact****.*

(2) A homomorphism $f : D \to N$ *lifts to a homomorphism* $F : D \to M$ *if and only if* f *is in the image of* φ_**. In general* φ_* *need not be surjective;* φ_* *is surjective (in which case the sequence (10.10) can be extended to a short exact sequence) if and only if every homomorphism from* D *to* N *lifts to a homomorphism from* D *to* M*.*

(3) The sequence (10.10) is exact for all R*-modules* D *if and only if*

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \qquad \textit{is exact}.$$

Proof. (1) The only item in the first statement that has not already been proved is the exactness of (10.10) at $\operatorname{Hom}_R(D, M)$, i.e., $\ker \varphi_* = \operatorname{image} \psi_*$.

Suppose $F : D \to M$ is an element of $\operatorname{Hom}_R(D, M)$ lying in the kernel of φ_*, i.e., with $\varphi \circ F = 0$ as homomorphisms from D to N. If $d \in D$ is any element of D, this implies that $\varphi(F(d)) = 0$ and $F(d) \in \ker \varphi$. By the exactness of the sequence defining the extension M we have $\ker \varphi = \operatorname{image} \psi$, so there is some element $l \in L$ with $F(d) = \psi(l)$. Since ψ is injective, the element l is unique, so this gives a well defined map $F' : D \to L$ given by $F'(d) := l$. It is an easy check to verify that F' is a homomorphism, i.e., $F' \in \operatorname{Hom}_R(D, L)$. Since $\psi \circ F'(d) = \psi(l) = F(d)$ (for any $d \in D$), we have $F = \psi_*(F')$, which shows that F is in the image of ψ_*, proving that $\ker \varphi_* \subseteq \operatorname{image} \psi_*$.

Conversely, if F is in the image of ψ_* then $F = \psi_*(F')$ for some $F' \in \operatorname{Hom}_R(D, L)$, and so $\varphi(F(d)) = \varphi(\psi(F'(d)))$ for any $d \in D$. Since $\ker \varphi = \operatorname{image} \psi$ we have $\varphi \circ \psi = 0$, and it follows that $\varphi(F(d)) = 0$ for any $d \in D$, i.e., $\varphi_*(F) = 0$. Hence F is in the kernel of φ_*, proving the reverse containment: $\operatorname{image} \psi_* \subseteq \ker \varphi_*$.

(3) For the last statement in the theorem, note first that the surjectivity of φ was not required for the proof that (10.10) is exact, so the "if" portion of the statement has already been proved.

For the converse, suppose that the sequence (10.10) is exact for all R-modules D. In general, $\operatorname{Hom}_R(R, X) \cong X$ for any left R-module X, the isomorphism being given by $f \mapsto f(1_R)$. Taking $D = R$ in (10.10), the exactness of the sequence $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N$ follows easily. □

By the above theorem, the sequence

$$0 \longrightarrow \operatorname{Hom}_R(D, L) \xrightarrow{\psi_*} \operatorname{Hom}_R(D, M) \xrightarrow{\varphi_*} \operatorname{Hom}_R(D, N) \longrightarrow 0 \tag{10.11}$$

is in general not a short exact sequence since the homomorphism φ_* need not be surjective. The next result characterizes the modules D having the property that the sequence (10.10) can always be extended to a short exact sequence.

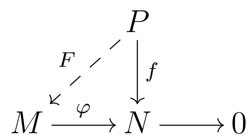
Proposition ([1] §10.5 Proposition 30 & Corollary 32). *Let P be an R-module. Then TFAE (the following are equivalent). The R-module P is called* ***projective*** *if it satisfies any of the following equivalent conditions.*

(1) For any R-modules L, M, and N, if $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ is a short exact sequence, then

$$0 \longrightarrow \operatorname{Hom}_R(P, L) \xrightarrow{\psi_*} \operatorname{Hom}_R(P, M) \xrightarrow{\varphi_*} \operatorname{Hom}_R(P, N) \longrightarrow 0$$

is also a short exact sequence, i.e. the functor $\operatorname{Hom}_R(P, \_)$ is ***exact*** *(it always takes short exact sequences to short exact sequences).*

(2) (The universal lifting property) *For any R-modules M and N, if $M \xrightarrow{\varphi} N \to 0$ is exact, then every R-module homomorphism from P into N lifts to an R-module homomorphism into M, i.e., given $f \in \operatorname{Hom}_R(P, N)$ there is a lift $F \in \operatorname{Hom}_R(P, M)$ making the following diagram commute:*



(3) If P is a quotient of the R-module M, then P is isomorphic to a direct summand of M, i.e., every short exact sequence $0 \to L \to M \to P \to 0$ splits. (P is called "projective" because any module M that projects onto P has (an isomorphic copy of) P as a direct summand.)

(4) P is a direct summand of a free R-module.

Proof. The equivalence of (1) and (2) is a restatement of the above theorem. The others are left as exercises. □

Example. *Free modules are projective. For example, $\mathbb{Z}$ is a projective $\mathbb{Z}$-module.*

This can be seen directly as follows: suppose f is a map from $\mathbb{Z}$ to N and $M \xrightarrow{\varphi} N \to 0$ is exact. The homomorphism f is uniquely determined by the value $n := f(1)$. Then f can be lifted to a homomorphism $F : \mathbb{Z} \to M$ by first defining $F(1) := m$, where m is any element in M mapped to n by φ, and then extending F to all of $\mathbb{Z}$ by additivity.

Since $\mathbb{Z}$ is projective, if $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ is an exact sequence of $\mathbb{Z}$-modules, then

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, L) \xrightarrow{\psi_*} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, M) \xrightarrow{\varphi_*} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, N) \longrightarrow 0$$

is also an exact sequence. This can also be seen directly using the isomorphism $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, M) \cong M$ of abelian groups, which shows that the two exact sequences above are essentially the same.

Example. *Free $\mathbb{Z}$-modules have no nonzero elements of finite order, so no nonzero finite abelian group can be isomorphic to a submodule of a free module. It follows that no nonzero finite abelian group is a projective $\mathbb{Z}$-module.*

In particular we show that, for $n \geq 2$ the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ is not projective. It must be possible to find a short exact sequence which after applying the functor $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \_)$ is no longer exact on the right. One such sequence is $0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0$. It is easy to see that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$ and $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. Applying $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \_)$ to the short exact sequence above thus gives the sequence

$$0 \longrightarrow 0 \xrightarrow{(\times n)_*} 0 \xrightarrow{\pi_*} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

which is not exact at its only nonzero term.

Another example is $\mathbb{Q}/\mathbb{Z}$. Since it is a torsion $\mathbb{Z}$-module it is not a submodule of a free $\mathbb{Z}$-module, hence is not projective. Note also that the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \to 0$ does not split since $\mathbb{Q}$ contains no submodule isomorphic to $\mathbb{Q}/\mathbb{Z}$.

Exercise ([1] §10.5 Exercise 8)**.** *Show that the $\mathbb{Z}$-module $\mathbb{Q}$ is not projective.*

Over many nice rings (such as $\mathbb{Z}$, fields, division algebras) every projective module is in fact a free module. Here are some examples to show that this is not always the case:

Example. *Let R be the ring $R_1 \times R_2$ under componentwise addition and multiplication. Then $P_1 = R_1 \times 0$ and $P_2 = 0 \times R_2$ are projective because their sum is R, but P_1 is not free since $(0,1)P_1 = 0$. For example, this is true when R is the ring $\mathbb{Z}/(6) = \mathbb{Z}/(2) \times \mathbb{Z}/(3)$.*

Example ([1] §18.2 Proposition 6)**.** *Consider the ring $R = M_n(F)$ of $n \times n$ matrices over a field F, acting on the left on the column vector space $V = F^n$. As a left R-module, R is the direct sum of its columns, each of which is the left R-module V. Hence $R \cong V \oplus \cdots \oplus V$, and V is a projective R-module.*

Since any free R-module would have dimension dn^2 over F for some cardinal number d, and $\dim_F(V) = n$, V cannot possibly be free over R.

Example. *Suppose that R is any field F, any ring $M_n(F)$ of matrices, or any group ring FG of a finite group G (such that $\operatorname{char} F$ does not divide $|G|$). Then every R-module is projective (cf. [1] Chap.18 Theorem 1, Theorem 4 & Proposition 6).*

2.2 Injective modules and $\operatorname{Hom}_R(\_, D)$

If $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ is a short exact sequence of R-modules then, instead of considering maps from an R-module D into L or N and the extent to which these determine maps from D into M, we can consider the "dual" question of maps from L or N to D.

In this case, it is easy to dispose of the situation of a map from N to D: an R-module map from N to D immediately gives a map from M to D simply by composing with φ. It is easy to check that this defines an injective homomorphism of abelian groups

$$\begin{aligned} \varphi^* : \operatorname{Hom}_R(N, D) &\longrightarrow \operatorname{Hom}_R(M, D) \\ f &\longmapsto f' := f \circ \varphi, \end{aligned}$$

or, put another way,

$$\text{if } M \xrightarrow{\varphi} N \longrightarrow 0 \text{ is exact,}$$

$$\text{then } \operatorname{Hom}_R(M, D) \xleftarrow{\varphi^*} \operatorname{Hom}_R(N, D) \longleftarrow 0 \text{ is also exact.}$$

(Note that the associated maps on the homomorphism groups are in the reverse direction from the original maps.)

If D is fixed, then given any R-module X we have an associated abelian group $\operatorname{Hom}_R(X, D)$. Further, an R-module homomorphism $\alpha : X \to Y$ induces an abelian group homomorphism $\alpha^* : \operatorname{Hom}_R(Y, D) \to \operatorname{Hom}_R(X, D)$, defined by $\alpha^*(f) := f \circ \alpha$, that "reverses" the direction of the arrow. Put another way, the map $\operatorname{Hom}_R(\_, D)$ is a **contravariant functor** from the category of R-modules to the category of abelian groups (cf. [1] Appendix II).

On the other hand, given an R-module homomorphism f from L to D, it may not be possible to extend f to a map F from M to D, i.e., given f it may not be possible to find a map F making the following diagram commute:

$$\begin{array}{ccc} L & \xrightarrow{\psi} & M \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle F} & \\ D & & \end{array}$$

Exercise. *Let $R = D$ be any field F. For any vector spaces $W \subseteq V$ over F, if $0 \to W \xrightarrow{\iota} V \xrightarrow{\pi} V/W \to 0$ is a short exact sequence, show that*

$$0 \longrightarrow (V/W)^* \xrightarrow{\pi^*} V^* \xrightarrow{\iota^*} W^* \longrightarrow 0$$

is also a short exact sequence. Describe the image of $(V/W)^*$ in V^*. Show in detail how to lift a linear functional of W to that of V (cf. [1] §10.5 Proposition 29, note that the dimensions might be infinite.) This exercise proves that F is an injective F-module.

Example. *Consider the nonsplit exact sequence $0 \to \mathbb{Z} \xrightarrow{\psi} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0$ of $\mathbb{Z}$-modules, where ψ is multiplication by 2. Let $D = \mathbb{Z}/2\mathbb{Z}$ and let $f\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ be reduction modulo 2 on the first $\mathbb{Z}$ in the sequence. There is only one nonzero homomorphism F from the second $\mathbb{Z}$ in the sequence to $\mathbb{Z}/2\mathbb{Z}$ (namely, reduction modulo 2), but this F does not lift the map f since $F \circ \psi(\mathbb{Z}) = F(2\mathbb{Z}) = 0$, so $F \circ \psi \neq f$.*

Composition with ψ induces a homomorphism ψ^* from $\mathrm{Hom}_R(M, D)$ to $\mathrm{Hom}_R(L, D)$ of abelian groups, and in terms of the map ψ^* the homomorphism $f \in \mathrm{Hom}_R(L, D)$ can be lifted to a homomorphism from M to D if and only if f is in the image of ψ^*. The example above shows that

$$\text{if } 0 \longrightarrow L \xrightarrow{\psi} M \text{ is exact,}$$

$$\text{then } 0 \longleftarrow \mathrm{Hom}_R(L, D) \xleftarrow{\psi^*} \mathrm{Hom}_R(M, D) \text{ is } \underline{\text{not necessarily}} \text{ exact.}$$

Theorem ([1] §10.5 Theorem 33 & Corollary 35)**.** *Let D, L, M, and N be R-modules.*

(1) If $(0 \to)L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ is exact, then the associated sequence

$$0 \longrightarrow \mathrm{Hom}_R(N, D) \xrightarrow{\varphi^*} \mathrm{Hom}_R(M, D) \xrightarrow{\psi_*} \mathrm{Hom}_R(L, D) \qquad \textit{is exact}, \tag{10.12}$$

*i.e., the functor $\mathrm{Hom}_R(\_, D)$ from the category of R-modules to the category of abelian groups is **left exact**.*

(2) A homomorphism $f : L \to D$ lifts to a homomorphism $F : M \to D$ if and only if f is in the image of ψ^*. In general ψ^* need not be surjective; ψ^* is surjective (in which case the sequence (10.12) can be extended to a short exact sequence) if and only if every homomorphism from L to D lifts to a homomorphism from M to D.

(3) The sequence (10.12) is exact for all R-modules D if and only if

$$L \xrightarrow{\psi} M \xrightarrow{\varphi} N \longrightarrow 0 \qquad \textit{is exact}.$$

Proof. Exercise. □

By the above theorem, the sequence

$$0 \longrightarrow \operatorname{Hom}_R(N, D) \xrightarrow{\varphi^*} \operatorname{Hom}_R(M, D) \xrightarrow{\psi^*} \operatorname{Hom}_R(L, D) \longrightarrow 0$$

is in general not a short exact sequence since the homomorphism ψ^* need not be surjective. The next result characterizes the modules D having the property that the sequence (10.12) can always be extended to a short exact sequence.

Proposition ([1] §10.5 Proposition 34 & Corollary 35)**.** *Let Q be an R-module. Then TFAE (the following are equivalent). The R-module Q is called* ***injective*** *if it satisfies any of the following equivalent conditions.*

(1) For any R-modules L, M, and N, if $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ is a short exact sequence, then

$$0 \longrightarrow \operatorname{Hom}_R(N, Q) \xrightarrow{\varphi^*} \operatorname{Hom}_R(M, Q) \xrightarrow{\psi^*} \operatorname{Hom}_R(L, Q) \longrightarrow 0$$

is also a short exact sequence, i.e. the functor $\operatorname{Hom}_R(\_, Q)$ is ***exact****.*

(2) (The universal lifting property) *For any R-modules L and M, if $0 \to L \xrightarrow{\psi} M$ is exact, then every R-module homomorphism from L into Q lifts to an R-module homomorphism from M into Q, i.e., given $f \in \operatorname{Hom}_R(L, Q)$ there is a lift $F \in \operatorname{Hom}_R(M, Q)$ making the following diagram commute:*

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \xrightarrow{\psi} & M \\ & & {\scriptstyle f}\downarrow & \swarrow {\scriptstyle F} & \\ & & Q & & \end{array}$$

(3) If Q is a submodule of the R-module M, then Q is a direct summand of M, i.e., every short exact sequence $0 \to Q \to M \to N \to 0$ splits. (Q is called "injective" because any module M into which Q injects has (an isomorphic copy of) Q as a direct summand.)

Proof. Exercise. □

Example. *$\mathbb{Z}$ is not an injective $\mathbb{Z}$-module. This follows from the fact that the exact sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ corresponding to multiplication by 2 does not split.*

We have seen that an R-module is projective if and only if it is a direct summand of a free R-module. Providing such a simple characterization of injective R-modules is not so easy. The next result gives a criterion for Q to be an injective R-module, and using it we can give a characterization of injective modules when $R = \mathbb{Z}$ (or, more generally, when R is a P.I.D.).

Proposition ([1] §10.5 Proposition 36)**.** *Let Q be an R-module.*

(1) (Baer's Criterion) *Q is injective if and only if, for every left ideal I of R, any R-module homomorphism $g : I \to Q$ can be extended to an R-module homomorphism $G : R \to Q$.*

(2) If R is a P.I.D. then Q is injective if and only if it is ***divisible****, i.e., $rQ = Q$ for every $0 \neq r \in R$.*

(3) When R is a P.I.D., quotient modules of injective R-modules are again injective.

Example. *Both $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible, therefore they are both injective $\mathbb{Z}$-modules. On the other hand, $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ are not divisible, and hence not injective.*

Example. *Every R-module is injective if R is any field F.*

Moreover, suppose that R is the ring $M_n(F)$ of matrices, or the group ring FG of a finite group G (such that $\operatorname{char} F$ does not divide $|G|$). Then every R-module is injective (cf. [1] Chap.18 Theorem 1, Theorem 4 & Proposition 6).

3 Flat modules and $D \otimes_R \_$

We now consider the behavior of extensions $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ of R-modules with respect to tensor products.

Theorem ([1] §10.4 Theorem 13)**.** *Let M, M' be right R-modules, let N, N' be left R-modules, and suppose $\mathcal{A} : M \to M'$, $\mathcal{B} : N \to N'$ are R-module homomorphisms.*

(1) There is a unique group homomorphism $\mathcal{C} : M \otimes_R N \to M' \otimes_R N'$, such that

$$\mathcal{C}(m \otimes n) = \mathcal{A}m \otimes \mathcal{B}n, \quad \text{for all } m \in M,\ n \in N.$$

Denote $\mathcal{C}$ by $\mathcal{A} \otimes \mathcal{B}$. If M, M' are also (S, R)-bimodules for some ring S and $\mathcal{A}$ is also an S-module homomorphism, then $\mathcal{A} \otimes \mathcal{B}$ is a homomorphism of left S-modules. In particular, if R is commutative then $\mathcal{A} \otimes \mathcal{B}$ is always an R-module homomorphism for the standard R-module structures.

(2) If $\mathcal{A}' : M' \to M''$, $\mathcal{B}' : N' \to N''$ are R-module homomorphisms, then

$$(\mathcal{A}' \otimes \mathcal{B}') \circ (\mathcal{A} \otimes \mathcal{B}) = (\mathcal{A}' \circ \mathcal{A}) \otimes (\mathcal{B}' \circ \mathcal{B}).$$

Suppose that D is a right R-module. For any homomorphism $f : X \to Y$ of left R-modules we obtain (via the above theorem) a homomorphism $\mathrm{id} \otimes f : D \otimes_R X \to D \otimes_R Y$ of abelian groups. If in addition D is an (S, R)-bimodule (for example, when $S = R$ is commutative and D is given the standard (R, R)-bimodule structure), then $\mathrm{id} \otimes f$ is a homomorphism of left S-modules.

Put another way, $D \otimes_R \_ : X \mapsto D \otimes_R X$ is a covariant functor from the category of left R-modules to the category of abelian groups (respectively, to the category of left S-modules when D is an (S, R)-bimodule). In a similar way, if D is a left R-module then $\_ \otimes_R D$ is a covariant functor from the category of right R-modules to the category of abelian groups (respectively, to the category of right S-modules when D is an (R, S)-bimodule).

Note that, unlike Hom, the tensor product is covariant in both variables, and we shall therefore concentrate on $D \otimes_R \_$, leaving as an exercise the minor alterations necessary for $\_ \otimes_R D$.

We have already seen examples where the map $\mathrm{id} \otimes \psi : D \otimes_R L \to D \otimes_R M$ induced by an injective map $\psi : L \to M$ is no longer injective (for example, the injection $\mathbb{Z} \hookrightarrow \mathbb{Q}$ of $\mathbb{Z}$-modules induces the zero map from $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$).

On the other hand, suppose that $\varphi : M \to N$ is a surjective R-module homomorphism. The tensor product $D \otimes_R N$ is generated as an abelian group by the simple tensors $d \otimes n$ for $d \in D$ and $n \in N$. The surjectivity of φ implies that $n = \varphi(m)$ for some $m \in M$, and then $(\mathrm{id} \otimes \varphi)(d \otimes m) = d \otimes \varphi(m) = d \otimes n$ shows that $\mathrm{id} \otimes \varphi$ is a surjective homomorphism of abelian groups from $D \otimes_R M$ to $D \otimes_R N$. This proves most of the following theorem.

Theorem ([1] §10.5 Theorem 39 & Corollary 41)**.** *Suppose that D is a right R-module and that L, M and N are left R-modules.*

(1) If $(0 \to) L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ is exact, then the associated sequence

$$D \otimes_R L \xrightarrow{\mathrm{id} \otimes \psi} D \otimes_R M \xrightarrow{\mathrm{id} \otimes \varphi} D \otimes_R N \longrightarrow 0 \qquad \text{is exact,} \tag{10.13}$$

*i.e., the functor $D \otimes_R \_$ from the category of left R-modules to the category of abelian groups is **right exact**.*

If D is an (S, R)-bimodule (for example when $S = R$ is commutative and D is given the standard R-module structure), then (13) is an exact sequence of left S-modules, and $D \otimes_R \_$ is a right exact functor from the category of left R-modules to the category of left S-modules.

(2) The map $\operatorname{id} \otimes \psi$ is not in general injective, i.e., the sequence (10.13) cannot in general be extended to a short exact sequence.

(3) The sequence (10.13) is exact for all R-modules D if and only if

$$0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\varphi} N \qquad \text{is exact.}$$

Proof. Exercise. □

The following result relating to modules D having the property that (10.13) can always be extended to a short exact sequence is immediate from the above theorem:

Proposition ([1] §10.5 Proposition 40 & Corollary 41)**.** *Let A be a right R-module. Then TFAE (the following are equivalent). A is called **flat** if it satisfies any of the following.*

(1) For any left R-modules L, M, and N, if $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ is a short exact sequence, then

$$0 \longrightarrow A \otimes_R L \xrightarrow{\operatorname{id} \otimes \psi} A \otimes_R M \xrightarrow{\operatorname{id} \otimes \varphi} A \otimes_R N \longrightarrow 0$$

*is also a short exact sequence, i.e. the functor $A \otimes_R \_$ is **exact**.*

(2) For any left R-modules L and M, if $0 \to L \xrightarrow{\psi} M$ is an exact sequence of left R-modules (i.e., $\psi : L \to M$ is injective), then $0 \to A \otimes_R L \xrightarrow{\operatorname{id} \otimes \psi} A \otimes_R M$ is an exact sequence of abelian groups (i.e., $\operatorname{id} \otimes \psi : A \otimes_R L \to A \otimes_R M$ is injective).

Example. *$\mathbb{Z}/2\mathbb{Z}$ is not a flat $\mathbb{Z}$-module. Applying $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \_$ to the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \to 0$ gives the sequence $0 \to \mathbb{Z}/2\mathbb{Z} \to 0 \to 0 \to 0$, which is not exact at its only nonzero term.*

Corollary ([1] §10.5 Corollary 42)**.** *Free modules are flat; more generally, projective modules are flat. (For example, $\mathbb{Z}$ is a flat $\mathbb{Z}$-module; any vector space over a field F is a flat F-module.)*

Proof. Here we only show that any finitely generated free R-module $F \cong R^n$ is flat. It suffices to show that, for any injective map $\psi : L \to M$ of R-modules L and M, the induced map $\operatorname{id} \otimes \psi : F \otimes_R L \to F \otimes_R M$ is also injective.

Recall that $R \otimes_R L \cong L$ and tensor products commute with direct sums (cf. [1] §10.4 Theorem 17). Therefore $F \otimes_R L \cong L^n$, $F \otimes_R M \cong M^n$, and under these isomorphisms the map $\operatorname{id} \otimes \psi : F \otimes_R L \to F \otimes_R M$ is just the natural map of L^n to M^n induced by ψ in each component. In particular, $\operatorname{id} \otimes \psi$ is injective and it follows that any finitely generated free module is flat. □

Example. *The $\mathbb{Z}$-module $\mathbb{Q}$ is a flat $\mathbb{Z}$-module. This shows that flat modules need not be projective.*

Example. *The $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is injective, but is not flat: if we identify $\mathbb{Q}/\mathbb{Z} = \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}$, then applying $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \_$ to the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0$ gives the sequence $0 \to \mathbb{Q}/\mathbb{Z} \xrightarrow{\times 2} \mathbb{Q}/\mathbb{Z} \to 0 \to 0$, which is not exact since multiplication by 2 has the element $1/2$ in its kernel.*

Exercise ([1] §10.5 Theorem 43, Adjoint Associativity)**.** *Let R and S be rings, let A be a right R-module, let B be an (R,S)-bimodule and let C be a right S-module. Show that there is an isomorphism of abelian groups:*

$$\operatorname{Hom}_S(A \otimes_R B, C) \cong \operatorname{Hom}_R(A, \operatorname{Hom}_S(B, C))$$

(the homomorphism groups are right module homomorphisms).

In particular, this exercise shows that $(V \otimes W)^* \cong \operatorname{Hom}_F(V, W^*)$, where V and W are vector spaces over F.

Summary

Each of the functors $\mathrm{Hom}_R(A, \_)$, $\mathrm{Hom}_R(\_, A)$, and $A \otimes_R \_$, map left R-modules to abelian groups; the functor $\_ \otimes_R A$ maps right R-modules to abelian groups. When R is commutative all four functors map R-modules to R-modules.

(1) Let A be a left R-module. The functor $\mathrm{Hom}_R(A, \_)$ is covariant and left exact; the module A is projective if and only if $\mathrm{Hom}_R(A, \_)$ is exact (i.e., is also right exact).

(2) Let A be a left R-module. The functor $\mathrm{Hom}_R(\_, A)$ is contravariant and left exact; the module A is injective if and only if $\mathrm{Hom}_R(\_, A)$ is exact.

(3) Let A be a right R-module. The functor $A \otimes_R \_$ is covariant and right exact; the module A is flat if and only if $A \otimes_R \_$ is exact (i.e., is also left exact).

(4) Let A be a left R-module. The functor $\_ \otimes_R A$ is covariant and right exact; the module A is flat if and only if $\_ \otimes_R A$ is exact.

(5) Projective modules are flat. The $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is injective but not flat. The $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Q}$ is flat but neither projective nor injective.

Other related exercises in [1]

§10.5 1 2 3 4 5 6 7 10 11 14 15 20 22 23 24 25 26

References

[1] D. S. Dummit and R. M. Foote. *Abstract algebra*. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004.