Shandong University Advanced Modern Algebra

Lecture 14: Introduction to Homological Algebra 2

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This lecture refers to Chapter 17 in [1]. All the equation numbers without reference labels are from this book.

1 Cohomology and homology groups

Cohomology and homology groups occur in many areas of mathematics. The formal notions of homology and cohomology groups and the general area of homological algebra arose from algebraic topology around the middle of the 20^{th} century in the study of the relation between the higher homotopy groups and the fundamental group of a topological space. Much of the language of homology and cohomology reflects its topological origins: homology groups, chains, cycles, boundaries, etc.

1.1 Definitions

We begin with a generalization of the notion of an exact sequence, namely a sequence of abelian group homomorphisms where successive maps compose to zero (i.e., the image of one map is contained in the kernel of the next):

Definition. *Let $\mathcal{C}$ be a sequence of abelian group homomorphisms:*

$$0 \longrightarrow C^0 \xrightarrow{d_1} C^1 \longrightarrow \cdots \longrightarrow C^{n-1} \xrightarrow{d_n} C^n \xrightarrow{d_{n+1}} \cdots .$$

*The sequence $\mathcal{C}$ is called a **cochain complex** if the composition of any two successive maps is zero: $d_{n+1} \circ d_n = 0$ for all n. The n^{th} **cohomology group** of a cochain complex $\mathcal{C}$ is the quotient group $\ker d_{n+1}/\operatorname{image} d_n$, and is denoted by $H^n(\mathcal{C})$.*

There is a completely analogous "dual" version in which the homomorphisms are between groups in decreasing order: if a sequence $\mathcal{C}$ of abelian group homomorphisms

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

*satisfies that the composition of any two successive homomorphisms is zero, then the complex $\mathcal{C}$ is called a **chain complex**, and its **homology groups** are defined as $H_n(\mathcal{C}) := \ker \partial_n/\operatorname{image} \partial_{n+1}$.*

For chain complexes the notation is often chosen so that the indices appear as subscripts and are *decreasing*, whereas for cochain complexes the indices are superscripts and are *increasing*. Let R be a commutative ring with 1. We shall also be interested in the situation where each C^n (respectively C_n) is an R-module and the homomorphisms d_n (respectively ∂_n) are R-module homomorphisms (referred to simply as a **complex of R-modules**), in which case the groups $H^n(\mathcal{C})$ (respectively $H_n(\mathcal{C})$) are also R-modules.

Note that if $\mathcal{C}$ is a cochain (respectively, chain) complex, then $\mathcal{C}$ is an exact sequence if and only if all its cohomology (respectively, homology) groups are zero. Thus the n^{th} cohomology (respectively, homology) group measures the failure of exactness of a complex at the n^{th} stage.

In this section we shall concentrate on cochains and cohomology, although all of the general results in this section have similar statements for chains and homology.

Let $\mathcal{A} = \{A^n\}$, $\mathcal{B} = \{B^n\}$ and $\mathcal{C} = \{C^n\}$ be cochain complexes. A **homomorphism of complexes** $\alpha : \mathcal{A} \to \mathcal{B}$ is a set of homomorphisms $\alpha_n : A^n \to B^n$ such that for every n the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha_{n+1}} & & \\ \cdots & \longrightarrow & B^n & \longrightarrow & B^{n+1} & \longrightarrow & \cdots \end{array} \tag{17.4}$$

A **short exact sequence** of complexes $0 \to \mathcal{A} \stackrel{\alpha}{\to} \mathcal{B} \stackrel{\beta}{\to} \mathcal{C} \to 0$ is a sequence of homomorphisms of complexes such that $0 \to A^n \stackrel{\alpha_n}{\to} B^n \stackrel{\beta_n}{\to} C^n \to 0$ is short exact for every n.

Proposition ([1] §17.1 Proposition 1)**.** *A homomorphism $\alpha : \mathcal{A} \to \mathcal{B}$ of cochain complexes induces group homomorphisms from $H^n(A)$ to $H^n(B)$ for $n \geq 0$ on their respective cohomology groups.*

Theorem ([1] §17.1 Theorem 2, the Long Exact Sequence in Cohomology)**.** *Let $0 \to \mathcal{A} \stackrel{\alpha}{\to} \mathcal{B} \stackrel{\beta}{\to} \mathcal{C} \to 0$ be a short exact sequence of cochain complexes. Then there is a long exact sequence of cohomology groups:*

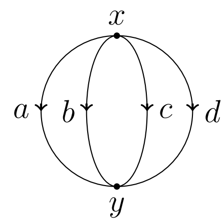
$$\begin{array}{l} 0 \longrightarrow H^0(\mathcal{A}) \longrightarrow H^0(\mathcal{B}) \longrightarrow H^0(\mathcal{C}) \stackrel{\delta_0}{\longrightarrow} \\ \quad H^1(\mathcal{A}) \longrightarrow H^1(\mathcal{B}) \longrightarrow H^1(\mathcal{C}) \stackrel{\delta_1}{\longrightarrow} \\ \quad H^2(\mathcal{A}) \longrightarrow H^2(\mathcal{B}) \longrightarrow H^2(\mathcal{C}) \longrightarrow \cdots \\ \quad \cdots \longrightarrow H^n(\mathcal{A}) \longrightarrow H^n(\mathcal{B}) \longrightarrow H^n(\mathcal{C}) \end{array} \tag{17.5}$$

*where the maps between cohomology groups at each level are those in the above proposition. The maps δ_n are called **connecting homomorphisms**.*

One immediate consequence of the existence of the long exact sequence in the above theorem is the fact that, if any two of the cochain complexes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are exact, then so is the third (cf. [1] §17.1 Exercise 6).

1.2 The idea of homology

Homology groups, by contrast, are quite directly related to cell structures, and may indeed be regarded as simply an algebraization of the first layer of geometry in cell structures: how cells of dimension n attach to cells of dimension $n-1$. Let us look at some examples (cf. [2] Chapter 2) to see what the idea is.



Consider the graph X_1 shown in the above figure, consisting of two vertices joined by four edges. When studying the fundamental group of X_1 we consider loops formed by sequences of edges, starting and ending at a fixed basepoint. For example, at the basepoint x, the loop ab^{-1} travels forward along the edge a, then backward along b, as indicated by the exponent -1. A salient feature of the fundamental group is that it is generally nonabelian, which both enriches and complicates the theory.

Suppose we simplify matters by abelianizing. Thus for example the two loops ab^{-1} and $b^{-1}a$ are to be regarded as equal. These two loops ab^{-1} and $b^{-1}a$ are really the same circle, just with a different choice of starting and ending point: x for ab^{-1} and y for $b^{-1}a$. The same thing happens for all loops: Rechoosing the basepoint in a loop just permutes its letters cyclically, so a byproduct of abelianizing is that we no longer have to pin all our loops down to a fixed basepoint. Thus loops become **cycles**, without a chosen basepoint.

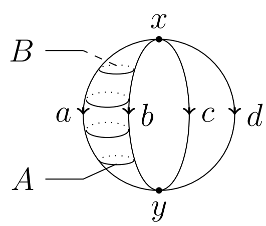
Having abelianized, let us switch to additive notation, so cycles become linear combinations of edges with integer coefficients, such as $a-b+c-d$. Let us call these linear combinations **chains** of edges. Some chains can be decomposed into cycles in several different ways, for example $(a-c)+(b-d)=(a-d)+(b-c)$, and if we adopt an algebraic viewpoint then we do not want to distinguish between these different decompositions. Thus we broaden the meaning of the term "cycle" to be simply any linear combination of edges for which at least one decomposition into cycles in the previous more geometric sense exists.

What is the condition for a chain to be a cycle in this more algebraic sense? A geometric cycle, thought of as a path traversed in time, is distinguished by the property that it enters each vertex the same number of times that it leaves the vertex. For an arbitrary chain $ka+lb+mc+nd$, the net number of times this chain enters y is $k+l+m+n$ since each of a, b, c, and d enters y once. Similarly, each of the four edges leaves x once, so the net number of times the chain $ka+lb+mc+nd$ enters x is $-k-l-m-n$. Thus the condition for $ka+lb+mc+nd$ to be a cycle is simply $k+l+m+n=0$.

To describe this result in a way that would generalize to all graphs, let C_1 be the free abelian group with basis the edges a, b, c, d and let C_0 be the free abelian group with basis the vertices x, y. Elements of C_1 are linear combinations of edges, or 1-dimensional chains, and elements of C_0 are linear combinations of vertices, or 0-dimensional chains. Define a homomorphism $\partial : C_1 \to C_0$ by sending each basis element a, b, c, d to $y-x$, the vertex at the head of the edge minus the vertex at the tail. Thus we have

$$\partial(ka+lb+mc+nd)=(k+l+m+n)y-(k+l+m+n)x,$$

and the cycles are precisely the kernel of ∂. It is a simple calculation to verify that every cycle in X_1 is a unique linear combination of three most obvious cycles $a-b$, $b-c$, and $c-d$. By means of these three basic cycles we convey the geometric information that the graph X_1 has three visible "holes", the empty spaces between the four edges.



Let us now enlarge the preceding graph X_1 by attaching two 2-cells A and B along the cycle $a - b$, producing a 2-dimensional cell complex X_2. If we think of the 2-cells as being oriented clockwise, then we can regard their boundaries as the cycle $a - b$. This cycle is now **homotopically** trivial since we can contract it to a point by sliding over A. In other words, it no longer encloses a hole in X_2. This suggests that we form a quotient of the group of cycles in the preceding example by factoring out the subgroup generated by $a - b$. In this quotient the cycles $a - c$ and $b - c$, for example, become equivalent, consistent with the fact that they are homotopic in X_2.

Algebraically, we can define now a pair of homomorphisms $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$, where C_2 is the free abelian group generated by A and B, and $\partial_2(A) = \partial_2(B) := a - b$. The map ∂_1 is the boundary homomorphism in the previous example. The quotient group we are interested in is $\ker \partial_1 / \operatorname{image} \partial_2$, the 1-dimensional cycles modulo those that are boundaries, the multiples of $a - b$. This quotient group is the homology group $H_1(X_2)$. The previous example can be fit into this scheme too by taking C_2 to be zero since there are no 2 cells in X_1, so in this case $H_1(X_1) = \ker \partial_1 / \operatorname{image} \partial_2 = \ker \partial_1$, which as we saw was free abelian on three generators. In the present example, $H_1(X_2)$ is free abelian on two generators, $b - c$ and $c - d$, expressing the geometric fact that by filling in a 2-cell we have reduced the number of "holes" in our space from three to two.

Again by taking C_3 to be zero, we have $H_2(X_2) = \ker \partial_2 / \operatorname{image} \partial_3 = \ker \partial_2$ which is the cyclic group generated by $A - B$. Topologically, the cycle $A - B$ is the sphere formed by the cells A and B together with their common boundary circle. This spherical cycle detects the presence of a "3-dimensional hole" in X_2, the missing interior of the sphere.

It is clear what the general pattern of the examples is. For a cell complex X one has chain groups $C^n(X)$ which are free abelian groups with basis the n-cells of X, and there are boundary homomorphisms $\partial_n : C_n \to C_{n-1}$, in terms of which one defines the homology group $H_n(X) := \ker \partial_n / \operatorname{image} \partial_{n+1}$. The major difficulty is how to define ∂_n in general. For larger n, even if one restricts attention to cell complexes formed from polyhedral cells with nice attaching maps, there is still the matter of orientations to sort out.

The best solution to this problem seems to be to adopt an indirect approach. Arbitrary polyhedra can always be subdivided into special polyhedra called simplices (the triangle and the tetrahedron are the 2-dimensional and 3-dimensional instances) so there is no loss of generality, though initially there is some loss of efficiency, in restricting attention entirely to simplices. For simplices there is no difficulty in defining boundary maps or in handling orientations. So one obtains a homology theory, called **simplicial homology**, for cell complexes built from simplices. Still, this is a rather restricted class of spaces, and the theory itself has a certain rigidity that makes it awkward to work with.

The way around these obstacles is to step back from the geometry of spaces decomposed into simplices and to consider instead something which at first glance seems wildly more complicated, the collection of all possible continuous maps of simplices into a given space X. These maps generate tremendously large chain groups $C_n(X)$, but the quotients $H_n(X) := \ker \partial_n / \operatorname{image} \partial_{n+1}$, called **singular homology groups**, turn out to be much smaller, at least for reasonably nice spaces X. In fact the singular homology groups coincide with the homology groups from the cellular chains. Moreover, singular homology allows one to define these nice cellular homology groups for all cell complexes, and in particular to solve the problem of defining the boundary maps for cellular chains.

1.3 The de Rham cohomology

In this section we define the de Rham cohomology (cf. [3] §1.1), which is the most important diffeomorphism invariant of a manifold.

Let $x_1, \ldots, x_n$ be the linear coordinates on $\mathbb{R}^n$. We define Ω^* to be the (exterior) algebra over $\mathbb{R}$ generated by $dx_1, \ldots, dx_n$ with the relations

$$(dx_i)^2 = 0; \qquad dx_i dx_j = -dx_j dx_i, \quad i \neq j.$$

As a vector space over $\mathbb{R}$, $\Omega^* \cong \Lambda(\mathbb{R}^n)$ has basis

$$1, \quad dx_i, \quad dx_i dx_j \ (i<j), \quad dx_i dx_j dx_k \ (i<j<k), \quad \ldots, \quad dx_1 \cdots dx_n.$$

The C^∞ **differential forms** on $\mathbb{R}^n$ are elements of

$$\Omega^*(\mathbb{R}^n) := \{C^\infty \text{ functions on } \mathbb{R}^n\} \otimes_{\mathbb{R}} \Omega^*.$$

Thus, if ω is such a form, then ω can be uniquely written as $\sum f_{i_1 \cdots i_k} dx_{i_1} \cdots dx_{i_k}$ where the coefficients $f_{i_1 \cdots i_k}$ are C^∞ functions (we also write $\omega = \sum f_I dx_I$). The algebra $\Omega^*(\mathbb{R}^n)$ is naturally graded: $\Omega^*(\mathbb{R}^n) = \bigoplus_{q=0}^n \Omega^q(\mathbb{R}^n)$, where $\Omega^q(\mathbb{R}^n)$ consists of the C^∞ q-forms on $\mathbb{R}^n$.

There is a differential operator $d : \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n)$ defined as follows:

- if $f \in \Omega^0(\mathbb{R}^n)$, then $df := \sum \dfrac{\partial f}{\partial x_i} dx_i$;
- if $\omega = \sum f_I dx_I$, then $d\omega := \sum df_I dx_I$.

This d, called the **exterior differentiation**, is the ultimate abstract extension of the usual gradient, curl, and divergence of vector calculus on $\mathbb{R}^3$, as the example below partially illustrates.

Example ([3] §1.1 Example 1.2)**.** *On* $\mathbb{R}^3$, $\Omega^0(\mathbb{R}^3)$ *and* $\Omega^3(\mathbb{R}^3)$ *are each 1-dimensional, and* $\Omega^1(\mathbb{R}^3)$ *and* $\Omega^2(\mathbb{R}^3)$ *are each 3-dimensional over the* C^∞ *functions. On functions,*

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

On 1*-forms,*

$$d(f_1 dx + f_2 dy + f_3 dz) = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) dydz - \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) dxdz + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dxdy.$$

On 2*-forms,*

$$d(f_1 dydz + f_2 dxdz + f_3 dxdy) = \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dxdydz.$$

In summary,

$$\begin{aligned} d\,(0\text{-}forms) &= \text{ gradient,} \\ d\,(1\text{-}forms) &= \text{ curl,} \\ d\,(2\text{-}forms) &= \text{ divergence.} \end{aligned}$$

Exercise ([3] §1.1 Proposition 1.4)**.** *Show that* $d^2 = 0$. [Remark: This is basically a consequence of the fact that the mixed partials are equal.]

The complex $\Omega^*(\mathbb{R}^n)$ together with the differential operator d is called the **de Rham complex** on $\mathbb{R}^n$. The kernel of d are the **closed forms** and the image of d, the **exact forms**. The q^{th} **de Rham cohomology** of $\mathbb{R}^n$ is the vector space

$$H^q_{dR}(\mathbb{R}^n) := \big(\Omega^q(\mathbb{R}^n) \cap \ker d\big)/\big(\Omega^q(\mathbb{R}^n) \cap \operatorname{image} d\big) = \{\text{closed } q\text{-forms}\}/\{\text{exact } q\text{-forms}\}.$$

Note that all the definitions so far work equally well for any open subset U of $\mathbb{R}^n$; for instance,

$$\Omega^*(U) := \{C^\infty \text{ functions on } U\} \otimes_{\mathbb{R}} \Omega^*$$

So we may also speak of the **de Rham cohomology** $H^q_{dR}(U)$ of U.

Example ([3] §1.1 Example 1.5)**.** *In general*

$$H^q_{dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{if } q = 0, \\ 0 & \text{if } q \geq 1. \end{cases}$$

This result is called the Poincaré Lemma. We will show in particular that $H^1_{dR}(\mathbb{R}^1) = 0$*: if* $\omega = g(x)dx$ *is a* 1*-form, then by taking* $f(x) := \int_0^x g(u)du$ *we find that* $df = g(x)dx$*. Therefore every* 1*-form on* $\mathbb{R}^1$ *is exact.*

Exercise. *Show that* $H^1_{dR}(\mathbb{R}^2) = H^1_{dR}(\mathbb{R}^3) = H^2_{dR}(\mathbb{R}^3) = 0$.

Theorem (The de Rham Theorem)**.** *The de Rham cohomology* $H^n_{dR}(X)$ *of a smooth manifold* X *(without boundary) is isomorphic to its singular cohomology* $H^n(X, \mathbb{R})$.

2 Derived functors

2.1 The Ext cohomology groups

Let R be a commutative ring with 1. Recall that a short exact sequence $0 \to L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$ of R-modules gives rise to an exact sequence of abelian groups

$$0 \longrightarrow \operatorname{Hom}_R(N, D) \xrightarrow{\varphi^*} \operatorname{Hom}_R(M, D) \xrightarrow{\psi^*} \operatorname{Hom}_R(L, D) \tag{17.2}$$

for any R-module D, and that the homomorphism ψ^* is in general not surjective so this sequence cannot always be extended to a short exact sequence. Equivalently, homomorphisms from L to D cannot in general be lifted to homomorphisms from M into D. The Long Exact Sequence in Cohomology provides a method of extending some exact sequences in a natural way.

We try to produce a cochain complex whose first few cohomology groups in the long exact sequence (17.5) agree with the terms in (17.2). To do this we introduce the notion of a "resolution" of an R-module:

Definition. *Let* A *be any* R*-module. A* ***projective resolution*** *of* A *is an exact sequence*

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \longrightarrow 0$$

such that each P_i *is a projective R-module (recall that* P *is* ***projective*** *if and only if the functor* $\operatorname{Hom}_R(P, \_)$ *is exact, if and only if* P *is a direct summand of a free* R*-module).*

Example. *Every R-module has a projective resolution: Let P_0 be any free (hence projective) R-module on a set of generators of A and define an R-module homomorphism ϵ from P_0 onto A by [1] §10.3 Theorem 6. This begins the resolution $\epsilon : P_0 \to A \to 0$. The surjectivity of ϵ ensures that this sequence is exact. Next let P_1 be any free module mapping onto the submodule $\ker \epsilon$ of P_0; this gives the second stage $P_1 \to P_0 \to A$ which, by construction, is also exact. We can continue this way, taking at the n^{th} stage a free R-module P_{n+1} that maps surjectively onto the submodule $\ker d_n$ of P_n, obtaining in fact a **free resolution** of A.*

In general a projective resolution is infinite in length, but if A is itself projective, then it has a very simple projective resolution of finite length, namely $0 \to A \xrightarrow{\text{id}} A \to 0$ given by the identity map from A to itself.

Given the projective resolution, we may form a related sequence by taking homomorphisms of each of the terms into an R-module D, keeping in mind that this reverses the direction of the homomorphisms. This yields the sequence

$$0 \longrightarrow \operatorname{Hom}_R(A, D) \xrightarrow{\epsilon} \operatorname{Hom}_R(P_0, D) \xrightarrow{d_1} \operatorname{Hom}_R(P_1, D) \xrightarrow{d_2} \cdots$$
$$\cdots \xrightarrow{d_{n-1}} \operatorname{Hom}_R(P_{n-1}, D) \xrightarrow{d_n} \operatorname{Hom}_R(P_n, D) \xrightarrow{d_{n+1}} \cdots$$

where to simplify notation we have denoted the induced maps $\operatorname{Hom}_R(P_{n-1}, D) \to \operatorname{Hom}_R(P_n, D)$ for $n \geq 1$ again by d_n and similarly for the map induced by ϵ. This sequence is not necessarily exact, however it is a cochain complex. The group

$$\operatorname{Ext}^n_R(A, D) := \begin{cases} \ker d_1 & \text{if } n = 0, \\ \ker d_{n+1} / \operatorname{image} d_n & \text{if } n > 0 \end{cases}$$

is called the n^{th} **cohomology group derived from the functor** $\operatorname{Hom}_R(\_, D)$. When $R = \mathbb{Z}$ the group $\operatorname{Ext}^n_{\mathbb{Z}}(A, D)$ is also denoted simply $\operatorname{Ext}^n(A, D)$.

Note that the groups $\operatorname{Ext}^n_R(A, D)$ are the cohomology groups of the cochain complex

$$0 \longrightarrow \operatorname{Hom}_R(P_0, D) \xrightarrow{d_1} \operatorname{Hom}_R(P_1, D) \xrightarrow{d_2} \operatorname{Hom}_R(P_2, D) \xrightarrow{d_3} \cdots$$

obtained by replacing $\operatorname{Hom}_R(A, D)$ with zero (which does not effect the cochain property).

Proposition ([1] §17.1 Proposition 3)**.** *For any R-module A we have $\operatorname{Ext}^0_R(A, D) \cong \operatorname{Hom}_R(A, D)$.*

Proof. Since the sequence $P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \to 0$ is exact, it follows that the corresponding sequence $0 \to \operatorname{Hom}_R(A, D) \xrightarrow{\epsilon} \operatorname{Hom}_R(P_0, D) \xrightarrow{d_1} \operatorname{Hom}_R(P_1, D)$ is also exact by [1] §10.5 Theorem 33. Hence $\operatorname{Ext}^0_R(A, D) = \ker d_1 = \operatorname{image} \epsilon \cong \operatorname{Hom}_R(A, D)$, as claimed. □

Example. *Let $R = \mathbb{Z}$ and let $A = \mathbb{Z}/m\mathbb{Z}$ for some $m \geq 2$. By the proposition we have*

$$\operatorname{Ext}^0_{\mathbb{Z}}(A, D) \cong \operatorname{Hom}_{\mathbb{Z}}(A, D) \cong {}_mD := \{d \in D \mid md = 0\},$$

the subgroup of D annihilated by m (containing elements of D with order dividing m).

For the higher cohomology groups, we use the simple projective resolution

$$0 \xrightarrow{d_2} \mathbb{Z} \xrightarrow[\times m]{d_1} \mathbb{Z} \xrightarrow{\epsilon} \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

for $A = \mathbb{Z}/m\mathbb{Z}$ given by multiplication by m on $\mathbb{Z}$. Taking homomorphisms into a fixed $\mathbb{Z}$-module D gives the cochain complex

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \xrightarrow{\epsilon} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, D) \xrightarrow[(\times m)^*]{d_1} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, D) \xrightarrow{d_2} 0 \xrightarrow{d_3} 0 \cdots .$$

Recall that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, D) \cong D$ *(defined by* $f \mapsto f(1)$*) and under this isomorphism we have*

$$\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) = \ker d_2 / \operatorname{image} d_1 \cong D/mD$$

for any abelian group D*. It follows from the definition and the cochain complex above that*

$$\operatorname{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) = 0 \quad \textit{for all } n \geq 2 \textit{ and any abelian group } D.$$

Example. *Let* P *be a projective* R*-module. Then the simple exact sequence* $0 \xrightarrow{d_1} P \underset{\text{id}}{\xrightarrow{\epsilon}} P \to 0$ *given by the identity map on* P *is a projective resolution of* P*. Taking homomorphisms into any* R*-module* D *gives the simple cochain complex*

$$0 \longrightarrow \operatorname{Hom}_R(P, D) \underset{\text{id}}{\xrightarrow{\epsilon}} \operatorname{Hom}_R(P, D) \xrightarrow{d_1} 0 \xrightarrow{d_2} 0 \cdots .$$

By definition

$$\operatorname{Ext}^n_R(P, D) = \begin{cases} \operatorname{Hom}_R(P, D) & \textit{if } n = 0, \\ 0 & \textit{if } n > 0. \end{cases}$$

In fact for an R*-module* P *the following are equivalent (cf. [1] §17.1 Proposition 11):*

(1) P *is projective,*

(2) $\operatorname{Ext}^1_R(P, D) = 0$ *for all* R*-modules* D*, and*

(3) $\operatorname{Ext}^n_R(P, D) = 0$ *for all* R*-modules* D *and all* $n \geq 1$*.*

The groups $\operatorname{Ext}^n_R(A, D)$ are independent of the choice of projective resolution of A (cf. [1] §17.1 Theorem 6). But the same abelian groups may be modules over several different rings R, and the Ext_R cohomology groups depend on R. For example, we have shown that $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \cong D/mD$ while $\operatorname{Ext}^1_{\mathbb{Z}/m\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) = 0$.

2.2 Right derived functors

Let $f : A \to A'$ be any homomorphism of R-modules and take projective resolutions of A and A', respectively. Then (cf. [1] §17.1 Proposition 4) for each $n \geq 0$ there is a lift f_n of f such that the following diagram commutes:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\ \cdots & \xrightarrow{d'_2} & P'_1 & \xrightarrow{d'_1} & P'_0 & \xrightarrow{\epsilon'} & A' & \longrightarrow & 0 \end{array}$$

where the rows are the projective resolutions of A and A', respectively. Then, the induced diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Hom}_R(A, D) & \xrightarrow{\epsilon} & \operatorname{Hom}_R(P_0, D) & \xrightarrow{d_1} & \operatorname{Hom}_R(P_1, D) & \xrightarrow{d_2} & \cdots \\ & & {\scriptstyle f^*}\uparrow & & {\scriptstyle f_0^*}\uparrow & & {\scriptstyle f_1^*}\uparrow & & \\ 0 & \longrightarrow & \operatorname{Hom}_R(A', D) & \xrightarrow{\epsilon'} & \operatorname{Hom}_R(P'_0, D) & \xrightarrow{d'_1} & \operatorname{Hom}_R(P'_1, D) & \xrightarrow{d'_2} & \cdots \end{array}$$

is also commutative. The two rows of this diagram are cochain complexes, and this commutative diagram depicts a homomorphism of these cochain complexes. By [1] §17.1 Proposition 1 we have an induced map on their cohomology groups: $\varphi_n : \operatorname{Ext}^n_R(A', D) \to \operatorname{Ext}^n_R(A, D)$, and the maps φ_n depend only on f, not on the choice of lifts f_n (cf. [1] §17.1 Proposition 5).

The next result shows that projective resolutions for a submodule and corresponding quotient module of an R-module M can be fit together to give a projective resolution of M.

Proposition ([1] §17.1 Proposition 7, Simultaneous Resolution)**.** *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules, let L and N both have projective resolutions where the projective modules are denoted by P_n and $\overline{P}_n$ respectively. Then there is a resolution of M by the projective modules $P_n \oplus \overline{P}_n$ such that the following diagram commutes:*

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1 & \longrightarrow & P_1 \oplus \overline{P}_1 & \longrightarrow & \overline{P}_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0 & \longrightarrow & P_0 \oplus \overline{P}_0 & \longrightarrow & \overline{P}_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Moreover, the rows and columns of this diagram are exact and the rows are split.

The above diagram induces a short exact sequence of cochain complexes

$$0 \longrightarrow \mathrm{Hom}_R(\overline{P}_\bullet, D) \longrightarrow \mathrm{Hom}_R(P_\bullet \oplus \overline{P}_\bullet, D) \longrightarrow \mathrm{Hom}_R(P_\bullet, D) \longrightarrow 0,$$

and next it induces a Long Exact Sequence in Cohomology (cf. [1] §17.1 Theorem 2):

Theorem ([1] §17.1 Theorem 8)**.** *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. Then there is a long exact sequence of abelian groups*

$$
\begin{array}{l}
0 \longrightarrow \mathrm{Hom}_R(N, D) \longrightarrow \mathrm{Hom}_R(M, D) \longrightarrow \mathrm{Hom}_R(L, D) \xrightarrow{\delta_0} \\
\quad \mathrm{Ext}^1_R(N, D) \longrightarrow \mathrm{Ext}^1_R(M, D) \longrightarrow \mathrm{Ext}^1_R(L, D) \xrightarrow{\delta_1} \\
\quad \mathrm{Ext}^2_R(N, D) \longrightarrow \quad \cdots
\end{array}
\tag{17.12}
$$

where the maps between groups at the same level n are as in [1] §17.1 Proposition 5 and the connecting homomorphisms δ_n are given by (17.5).

Exercise ([1] §17.1 Proposition 9)**.** *Show that, for an R-module Q the following are equivalent:*

(1) Q is injective,

(2) $\mathrm{Ext}^1_R(D, Q) = 0$ for all R-modules D, and

(3) $\mathrm{Ext}^n_R(D, Q) = 0$ for all R-modules D and all $n \geq 1$.

For a fixed R-module D, the result in the above theorem can be viewed as explaining what happens to the short exact sequence $0 \to L \to M \to N \to 0$ on the right after applying the left exact functor $\mathrm{Hom}_R(\_, D)$. This is why the (contravariant) functors $\mathrm{Ext}^n_R(\_, D)$ are called the **right derived functors** for the functor $\mathrm{Hom}_R(\_, D)$.

One can also consider the effect of applying the left exact functor $\mathrm{Hom}_R(D, \_)$, i.e., by taking homomorphisms from D rather than into D. The next theorem shows that in fact the same Ext_R groups define the (covariant) right derived functors for $\mathrm{Hom}_R(D, \_)$ as well.

Theorem ([1] §17.1 Theorem 10)**.** *Let* $0 \to L \to M \to N \to 0$ *be a short exact sequence of* R*-modules. Then there is a long exact sequence of abelian groups*

$$
\begin{aligned}
0 \longrightarrow \operatorname{Hom}_R(D, L) \longrightarrow \operatorname{Hom}_R(D, M) \longrightarrow \operatorname{Hom}_R(D, N) &\xrightarrow{\gamma_0} \\
\operatorname{Ext}^1_R(D, L) \longrightarrow \operatorname{Ext}^1_R(D, M) \longrightarrow \operatorname{Ext}^1_R(D, N) &\xrightarrow{\gamma_1} \\
\operatorname{Ext}^2_R(D, L) \longrightarrow \quad \cdots .&
\end{aligned}
\tag{17.14}
$$

2.3 Left derived functors

The cohomology groups $\operatorname{Ext}^n_R(A, B)$ determine what happens to short exact sequences on the right after applying the left exact functors $\operatorname{Hom}_R(D, \_)$ and $\operatorname{Hom}_R(\_, D)$. One may similarly ask for the behavior of short exact sequences on the left after applying the right exact functor $D \otimes_R \_$ or the right exact functor $\_ \otimes_R D$. This leads to the Tor (homology) groups (whose name derives from their relation to torsion submodules), and we now briefly outline the development of these left derived functors. In some respects this theory is "dual" to the theory for Ext_R.

We concentrate on the situation for $D \otimes_R \_$ when D is a right R-module. When D is a left R-module there is a completely symmetric theory for $\_ \otimes_R D$; when R is commutative and all R-modules have the same left and right R-action, the homology groups resulting from both developments are isomorphic.

Suppose then that D is a right R-module. Then for every left R-module B the tensor product $D \otimes_R B$ is an abelian group and the functor $D \otimes_R \_$ is covariant and right exact, i.e., for any short exact sequence $0 \to L \to M \to N \to 0$ of left R-modules,

$$D \otimes_R L \longrightarrow D \otimes_R M \longrightarrow D \otimes_R N \longrightarrow 0$$

is an exact sequence of abelian groups. This sequence may be extended at the left end to a long exact sequence as follows.

Let

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} B \longrightarrow 0$$

be a projective resolution of B. It follows from the argument in [1] §10.5 Theorem 39 that

$$\cdots \longrightarrow D \otimes P_n \xrightarrow{\operatorname{id} \otimes d_n} D \otimes P_{n-1} \longrightarrow \cdots \xrightarrow{\operatorname{id} \otimes d_1} D \otimes P_0 \xrightarrow{\operatorname{id} \otimes \epsilon} D \otimes B \longrightarrow 0$$

is a chain complex — the composition of any two successive maps is zero — so we may form its homology groups:

$$\operatorname{Tor}^R_n(D, B) := \begin{cases} (D \otimes P_0)/\operatorname{image}(\operatorname{id} \otimes d_1) & \text{if } n = 0, \\ \ker(\operatorname{id} \otimes d_n)/\operatorname{image}(\operatorname{id} \otimes d_{n+1}) & \text{if } n > 0 \end{cases}$$

The group $\operatorname{Tor}^R_n(D, B)$ is called the n^{th} **homology group derived from the functor** $D \otimes \_$. When $R = \mathbb{Z}$ the group $\operatorname{Tor}^{\mathbb{Z}}_n(D, B)$ is also denoted simply $\operatorname{Tor}_n(D, B)$.

Note that the groups $\operatorname{Tor}^R_n(D, B)$ is the n^{th} homology group of the chain complex

$$\cdots \longrightarrow D \otimes P_n \xrightarrow{\operatorname{id} \otimes d_n} D \otimes P_{n-1} \longrightarrow \cdots \xrightarrow{\operatorname{id} \otimes d_1} D \otimes P_0 \longrightarrow 0$$

obtained by removing the term $D \otimes B$.

Exercise ([1] §17.1 Proposition 13)**.** *Show that* $\operatorname{Tor}_0^R(D,B) = D \otimes_R B$ *for any left* R*-module* B *and any right* R*-module* D*.*

There is a Long Exact Sequence in Homology analogous to [1] §17.1 Theorem 2, except that all the arrows are reversed:

Theorem ([1] §17.1 Theorem 15)**.** *Let* $0 \to L \to M \to N \to 0$ *be a short exact sequence of left* R*-modules. Then there is a long exact sequence of abelian groups*

$$\cdots \longrightarrow \operatorname{Tor}_2^R(D,N) \xrightarrow{\delta_1} \operatorname{Tor}_1^R(D,L) \longrightarrow \operatorname{Tor}_1^R(D,M) \longrightarrow \operatorname{Tor}_1^R(D,N) \xrightarrow{\delta_0} D \otimes L \longrightarrow D \otimes M \longrightarrow D \otimes N \longrightarrow 0$$

We have defined $\operatorname{Tor}_n^R(A,B)$ as the homology of the chain complex obtained by tensoring a projective resolution of B on the left with A. The same groups are obtained by taking the homology of the chain complex obtained by tensoring a projective resolution of A on the right by B. Put another way, the $\operatorname{Tor}_n^R(A,B)$ groups define the (covariant) **left derived functors** for both of the right exact functors $A \otimes_R \_$ and $\_ \otimes_R B$.

Exercise ([1] §17.1 Proposition 9)**.** *Show that, for a right* R*-module* D *the following are equivalent:*

(1) D *is flat,*

(2) $\operatorname{Tor}_1^R(D,B) = 0$ *for all left* R*-modules* B*, and*

(3) $\operatorname{Tor}_n^R(D,B) = 0$ *for all left* R*-modules* B *and all* $n \geq 1$*.*

Finally, we mention that the cohomology and homology theories we have described may be developed in a vastly more general setting by axiomatizing the essential properties of R-modules and the Hom_R and tensor product functors. This leads to the general notions of **abelian categories** and **additive functors**.

In the case of the abelian category of R-modules, any additive functor $\mathcal{F}$ to the category of abelian groups gives rise to a set of derived functors, $\mathcal{F}_n$, also from R-modules to abelian groups, for all $n \geq 0$. Then for each short exact sequence $0 \to L \to M \to N \to 0$ of R-modules, there is a long exact sequence of (cohomology or homology) groups whose terms are $\mathcal{F}_n(L)$, $\mathcal{F}_n(M)$ and $\mathcal{F}_n(N)$, and these long exact sequences reflect the exactness properties of the functor $\mathcal{F}$. If $\mathcal{F}$ is left or right exact then the 0^{th} derived functor $\mathcal{F}_0$ is naturally equivalent to $\mathcal{F}$ (hence the 0^{th} degree groups $\mathcal{F}_0(X)$ are isomorphic to $\mathcal{F}(X)$), and if $\mathcal{F}$ is an exact functor then $\mathcal{F}_n(X) = 0$ for all $n \geq 1$ and all R-modules X.

3 The cohomology of groups

In this section we consider the application of the general techniques of the previous section in an important special case.

Let G be a group. An abelian group A on which G acts (on the left) as automorphisms is called a G**-module**. Note that a G-module is the same as an abelian group A and a homomorphism

$\varphi : G \to \mathrm{Aut}(A)$ of G into the group of automorphisms of A. Since an abelian group is the same as a module over $\mathbb{Z}$, it is also easy to see that a G-module A is the same as a $\mathbb{Z}G$-module over the integral group ring,

$$\mathbb{Z}G := \left\{ \sum\nolimits_{g \in G} \lambda_g g \mid \alpha_g \in \mathbb{Z},\ \alpha_g = 0 \text{ for all but finitely many } g \right\},$$

of G with coefficients in $\mathbb{Z}$.

As usual we shall often use multiplicative notation and write ga in place of $g \cdot a$ for the action of the element $g \in G$ on the element $a \in A$. Let

$$A^G := \{a \in A \mid ga = a \text{ for all } g \in G\}$$

be the set of elements of A fixed by all the elements of G. Then **fixed point subgroup** A^G is clearly a $\mathbb{Z}G$-submodule of A on which G acts trivially.

Example. *If K/F is an extension of fields that is Galois with Galois group G, then the additive group K is naturally a G-module, with $K^G = F$. Similarly, the multiplicative group $K^\times$ of nonzero elements in K is a* **G***-module, with fixed points $(K^\times)^G = F^\times$.*

It is easy to see that a short exact sequence $0 \to A \to B \to C \to 0$ of G-modules induces an exact sequence

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G$$

that in general cannot be extended to a short exact sequence (in general a coset in the quotient $C \cong B/A$ that is fixed by G need not be represented by an element in B fixed by G). One way to show the exactness is to observe that A^G can be related to a Hom group:

Exercise ([1] §17.2 Lemma 19)**.** *Suppose A is a G-module and $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$ is the group of all $\mathbb{Z}G$-module homomorphisms from $\mathbb{Z}$ (with* ***trivial*** *G-action, i.e., $ga = a$ for any $g \in G$, $a \in \mathbb{Z}$) to A. Then $f \mapsto f(1)$ defines an isomorphism $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A) \cong A^G$ of $\mathbb{Z}G$-modules.*

The above lemma also shows that, any projective resolution of $\mathbb{Z}$ (considered as a $\mathbb{Z}G$-module) will give a long exact sequence extending $0 \to A^G \to B^G \to C^G$. One such projective resolution is the **standard resolution** or **bar resolution** of $\mathbb{Z}$:

$$\cdots \longrightarrow F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \cdots \xrightarrow{d_1} F_0 \xrightarrow{\mathrm{aug}} \mathbb{Z} \longrightarrow 0. \tag{17.16}$$

Here $F_n := (\mathbb{Z}G)^{\otimes(n+1)} = \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}G$ (where there are $n+1$ factors) for $n \geq 0$, which is a G-module under the action defined on simple tensors by

$$g \cdot (g_0 \otimes g_1 \otimes \cdots \otimes g_n) := (gg_0) \otimes g_1 \otimes \cdots \otimes g_n.$$

It is not difficult to see that F_n is a free $\mathbb{Z}G$-module of rank $|G|^n$, with $\mathbb{Z}G$-basis given by the elements $1 \otimes g_1 \otimes g_2 \otimes \cdots \otimes g_n$, where $g_i \in G$. The map $\mathrm{aug} : F_0 \to \mathbb{Z}$ is the **augmentation map** $\mathrm{aug}(\sum_{g \in G} \alpha_g g) := \sum_{g \in G} \alpha_g$, and the map d_1 is given by $d_1(1 \otimes g) := g - 1$. The maps d_n for $n \geq 2$ are more complicated and their definition, together with a proof that (17.16) is a projective (in fact, free) resolution can be found in [1] §17.2 Exercises 1-3.

Applying ($\mathbb{Z}G$-module) homomorphisms from the terms in (17.16) to the G-module A (replacing the first term by 0) as in the previous section, we obtain the cochain complex

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(F_0, A) \xrightarrow{d_1} \mathrm{Hom}_{\mathbb{Z}G}(F_1, A) \xrightarrow{d_2} \mathrm{Hom}_{\mathbb{Z}G}(F_2, A) \xrightarrow{d_3} \cdots, \tag{17.17}$$

the cohomology groups of which are, by definition, the groups $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$. Then, as in [1] §17.1 Theorem 8, the short exact sequence $0 \to A \to B \to C \to 0$ of G-modules gives rise to a long exact sequence whose first terms are given by $0 \to A^G \to B^G \to C^G$ and whose higher terms are the cohomology groups $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$.

To make this more explicit, we can reinterpret the terms in this cochain complex without explicit reference to the standard resolution of $\mathbb{Z}$, as follows. The elements of $\mathrm{Hom}_{\mathbb{Z}G}(F_n, A)$ are uniquely determined by their values on the $\mathbb{Z}G$-basis elements of F_n, which may be identified with the n-tuples $(g_1, g_2, \dots, g_n)$ of elements $g_i \in G$. It follows for $n \geq 1$ that the group $\mathrm{Hom}_{\mathbb{Z}G}(F_n, A)$ may be identified with the set of functions from $G^n := G \times \cdots \times G$ (n copies) to A. For $n = 0$ we identify $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A)$ with A.

Definition. *If G is a finite group and A is a G-module, define*

$$C^n(G, A) := \begin{cases} \{\textit{all maps } f : \underbrace{G \times \cdots \times G}_{n \text{ copies}} \to A\} & \textit{if } n \geq 1, \\ A & \textit{if } n = 0. \end{cases}$$

*The elements of $C^n(G, A)$ are called n-**cochains** (of G with values in A). Each $C^n(G, A)$ is an additive abelian group given by the usual pointwise addition of functions (for $n \geq 1$).*

Under the identification of $\mathrm{Hom}_{\mathbb{Z}G}(F_n, A)$ with $C^n(G, A)$, the cochain maps d_n in (17.17) can be given very explicitly: for $n \geq 0$, $d_n : C^n(G, A) \to C^{n+1}(G, A)$ is given by

$$\begin{aligned} d_n(f)(g_1, \dots, g_{n+1}) := {} & g_1 \cdot f(g_2, \dots, g_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_{n+1}) \\ & + (-1)^{n+1} f(g_1, \dots, g_n), \end{aligned} \tag{17.18}$$

*where the product $g_i g_{i+1}$ occupying the i^{th} position of f is taken in the group G. It is immediate from the definition that the maps d_n are group homomorphisms, called the n^{th} **coboundary homomorphisms**.*

Define

$$Z^n(G, A) := \ker d_n \quad \textit{for } n \geq 0; \qquad B^n(G, A) := \begin{cases} 0 \in A & \textit{for } n = 0, \\ \mathrm{image}\, d_{n-1} & \textit{for } n \geq 1. \end{cases}$$

*The elements of $Z^n(G, A)$ are called n-**cocycles**, and the elements of $B^n(G, A)$ are called n-**coboundaries**. It follows from the fact that (17.17) is a projective resolution that $d_n \circ d_{n-1} = 0$ for $n \geq 1$, so that $B^n(G, A)$ is always a subgroup of $Z^n(G, A)$. The quotient group*

$$H^n(G, A) := Z^n(G, A)/B^n(G, A), \quad n \geq 0$$

*is called the n^{th} **cohomology group of G with coefficients in A**.*

The definition of the cohomology group $H^n(G, A)$ in terms of cochains will be particularly useful when we examine the low dimensional groups $H^1(G, A)$ and $H^2(G, A)$ and their application in a variety of settings. It should be remembered, however, that $H^n(G, A) \cong \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$ for all $n \geq 0$. In particular, these groups can be computed using any projective resolution of $\mathbb{Z}$.

Example. *For $f = a \in C^0(G, A)$ we have $d_0(f)(g) = g \cdot a - a$ and so*

$$H^0(G, A) = Z^0(G, A) = \ker d_0 = \{a \in A \mid g \cdot a = a \textit{ for all } g \in G\} = A^G$$

for any group G and G-module A.

For $f \in C^1(G, A)$ we have

$$d_1(f)(g_1, g_2) = g_1 \cdot f(g_2) - f(g_1 g_2) + f(g_1).$$

Thus any function $f : G \to A$ is a 1-cocycle if and only if it satisfies the identity

$$f(gh) = f(g) + g \cdot f(h) \quad \text{for all } g, h \in G. \tag{17.20}$$

By the definition of d_0, a 1-cochain f is a 1-coboundary if there is some $a \in A$ such that

$$f(g) = g \cdot a - a \quad \text{for all } g \in G. \tag{17.21}$$

One can verify directly that a 1-coboundary is always a 1-cocycle.

Exercise. *Write explicitly the conditions for a function $f : G \times G \to A$ being a 2-cocyle or a 2-coboundary, and verify directly that a 2-coboundary is always a 2-cocycle.*

Example (Cohomology of a Finite Cyclic Group)**.** *Suppose $G = \langle \sigma \rangle$ is cyclic of order m. Let $N := 1 + \sigma + \sigma^2 + \cdots + \sigma^{m-1} \in \mathbb{Z}G$. Then $N(\sigma - 1) = (\sigma - 1)N = \sigma^m - 1 = 0$, and so we have a particularly simple free resolution*

$$\cdots \xrightarrow{\sigma-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{\sigma-1} \cdots \xrightarrow{N} \mathbb{Z}G \xrightarrow{\sigma-1} \mathbb{Z}G \xrightarrow{\text{aug}} \mathbb{Z} \longrightarrow 0$$

where aug *denotes the augmentation map. Taking $\mathbb{Z}G$-module homomorphisms from the terms of this resolution to A (replacing the first term by 0) and using the identification* $\operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) = A$ *gives the chain complex*

$$0 \longrightarrow A \xrightarrow{\sigma-1} A \xrightarrow{N} A \xrightarrow{\sigma-1} A \xrightarrow{N} \cdots$$

whose cohomology computes the groups $H^n(G, A)$:

$$H^0(G, A) = A^G, \quad \text{and} \quad H^n(G, A) = \begin{cases} A^G/NA & \text{if } n \text{ is even, } n \geq 2 \\ {}_N A/(\sigma - 1)A & \text{if } n \text{ is odd, } n \geq 1 \end{cases}$$

where ${}_N A := \{a \in A \mid Na = 0\}$ is the subgroup of A annihilated by N, since the kernel of multiplication by $\sigma - 1$ is A^G.

If in particular $G = \langle \sigma \rangle$ acts trivially on A, then $N \cdot a = ma$, so that in this case $H^0(G, A) = A$, with $H^n(G, A) = A/mA$ for even $n \geq 2$; and $H^n(G, A) = {}_m A$, the elements of A of order dividing m, for odd $n \geq 1$.

Other related exercises in [1]

§17.1 1 2 4 5 7 9 12 13 14 18
§17.2 1 2 3 4 5 8 9
§17.3 1 13

References

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