Lecture 15: Representation Theory of Finite Groups

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This lecture refers to Chapters 18 & 19 in [1]. All the equation numbers without reference labels are from this book.

1 Introduction to representation theory

1.1 Definitions and examples

Throughout this section F is a field and G is a group. We first introduce the basic terminology. Recall that if V is a vector space over F, then $\mathrm{GL}(V)$ is the group of nonsingular linear transformations from V to itself (under composition), and if $n \in \mathbb{Z}_{>0}$, then $\mathrm{GL}_n(F)$ is the group of invertible $n \times n$ matrices with entries from F (under matrix multiplication).

Definition 1.1. *Let G be a group and F be a field. A* ***linear representation*** *(or a* ***representation****) of G is a vector space V over F together with a homomorphism $\varphi : G \to \mathrm{GL}(V)$. Equivalently, it is a set-theoretic action of G on V given by $g \cdot v := \varphi(g)v$, which preserves the linear structure:*

$$\varphi(g)(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \varphi(g) v_1 + \lambda_2 \varphi(g) v_2, \quad \text{for any } g \in G,\ \lambda_1, \lambda_2 \in F,\ v_1, v_2 \in V.$$

(If G is a topology group or a Lie group, φ is usually assumed to be continuous or smooth, respectively.) Sometimes we simply denote the representation by V if φ is clear from the context. The ***degree*** *(or* ***dimension****) of the representation is the dimension of V.*

Let $n \in \mathbb{Z}_{>0}$. A ***matrix representation*** *of G is any homomorphism from G into $\mathrm{GL}_n(F)$.*

When we wish to emphasize the field F we shall say F-representation, or representation of G on V over F.

Recall that if V is a finite dimensional vector space of dimension n, then by fixing a basis of V we obtain an isomorphism $\mathrm{GL}(V) \cong \mathrm{GL}_n(F)$. In this way any linear representation of G on a finite dimensional vector space gives a matrix representation and vice versa.

A linear or matrix representation is **faithful** if it is injective. In this case G is isomorphic to a subgroup of $\mathrm{GL}_n(F)$.

Example. *Consider F^1 as a 1-dimensional vector space over F and consider the trivial action of G on F^1 by letting $gv \coloneqq v$ for all $g \in G$ and $v \in F^1$. This action affords the representation $\varphi : G \to \mathrm{GL}(F^1)$ defined by $\varphi(g) \coloneqq \mathrm{id}$ = the identity linear transformation, for all $g \in G$. The corresponding matrix representation (with respect to any basis of V) is the homomorphism of G into $\mathrm{GL}_1(F)$ which sends every group element to the 1×1 identity matrix $[1]$. We shall henceforth refer to this as the **trivial representation** of G. The trivial representation has degree 1 and if $|G| > 1$, it is not faithful.*

Example. *Let V be a finite dimensional vector space over F, and G be a subgroup of $\mathrm{GL}(V)$. Then the embedding map $G \hookrightarrow \mathrm{GL}(V)$ naturally gives a representation of G on V, called the **standard representation** of G. For example, $SO(3)$ has a standard representation of degree 3 on the 3-dimensional Euclidean space $\mathbb{R}^3$.*

Let $\varphi : G \to \mathrm{GL}(V)$ be a representation. A subspace U of V is called G**-invariant** or G**-stable**, or a **subrepresentation** of V, if $g \cdot u \in U$ (i.e. $\varphi(g)(u) \in U$) for all $g \in G$ and all $u \in U$. The representation V is said to be **irreducible** if its only subrepresentations are 0 and V; otherwise V is called **reducible**. If U is G-invariant, then the G-action on V descends to an action on V/U by setting $g\cdot(v+U) \coloneqq \varphi(g)(v)+U$. (If we choose another v' in the same coset as v, then $vv' \in U$, so $\varphi(g)(vv') \in U$, and then the cosets $\varphi(g)(v) + U$ and $\varphi(g)(v') + U$ agree.)

Suppose V is a finite dimensional representation of G and V is reducible. Let U be a G-invariant subspace. Form a basis of V by taking a basis of U and enlarging it to a basis of V. Then for each $g \in G$ the matrix, $\varphi(g)$, of g acting on V with respect to this basis is of the form

$$\varphi(g) = \begin{bmatrix} \varphi_1(g) & \psi(g) \\ 0 & \varphi_2(g) \end{bmatrix}$$

where $\varphi_1 = \varphi|_U$ (with respect to the chosen basis of U) and φ_2 is the representation of G on V/U (and ψ is not necessarily a homomorphism — $\psi(g)$ need not be a square matrix). So *reducible representations are those with a corresponding matrix representation whose matrices are in block upper triangular form.*

Example. *Let $n \in \mathbb{Z}_{>0}$, let $G = S_n$ and let V be an n-dimensional vector space over F with basis $e_1, e_2, \ldots, e_n$. Let S_n act on V by defining for each $\sigma \in S_n$*

$$\sigma \cdot e_i \coloneqq e_{\sigma(i)}, \quad 1 \le i \le n,$$

i.e., σ acts by permuting the subscripts of the basis elements. This provides an (injective) homomorphism of S_n into $\mathrm{GL}(V)$ (i.e., a faithful representation of S_n of degree n). As in the preceding example, the matrix of σ with respect to the basis $e_1, \ldots, e_n$ has a 1 in row i and column j if $\sigma \cdot e_j = e_i$ (and has 0 in all other entries). Thus σ has a 1 in row i and column j if $\sigma(j) = i$. For example, when $n = 3$,

$$(1\ 2) \cdot [e_1\ e_2\ e_3] = [e_1\ e_2\ e_3] \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (1\ 2\ 3) \cdot [e_1\ e_2\ e_3] = [e_1\ e_2\ e_3] \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Let N be the subspace of V consisting of vectors all of whose coordinates are equal, i.e.,

$$N \coloneqq \{\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n \mid \lambda_1 = \lambda_2 = \cdots = \lambda_n\} = F(e_1 + e_2 + \cdots + e_n).$$

*This is a 1-dimensional S_n-stable subspace (and hence irreducible, called the **trace** submodule of FS_n). Each $\sigma \in S_n$ fixes each vector in N so it is the trivial representation of S_n. As an exercise, one may show that if $n \ge 3$ then N is the unique 1-dimensional subspace of V which is S_n-stable.*

Another S_n-stable subspace of V is the subspace I of all vectors whose coordinates sum to zero:

$$I \coloneqq \{\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n \mid \lambda_1 + \lambda_2 + \cdots + \lambda_n = 0\}.$$

Again I is S_n-stable since each $\sigma \in S_n$ permutes the coordinates of each vector in V, each σ leaves the sum of the coefficients unchanged. Since I is the kernel of the linear transformation $V \to F$, $\sum \lambda_i e_i \mapsto \sum \lambda_i$ (called the ***augmentation map****), I has dimension $n-1$. It is called the* ***standard representation*** *of S_n, and it is irreducible if* char F *does not divide $n!$.*

Exercise ([1] §18.2 page 862)**.** *Write explicitly the images (as 2×2 matrices) of every element in S_3 under the standard representation.*

Recall that in the study of a linear transformation T of a vector space V to itself we made V into an $F[x]$-module (where x acted as T on V); our goal was to decompose V into a direct sum of cyclic submodules. In this way we were able to find a basis of V for which the matrix of T with respect to this basis was in some *canonical* form. Changing the basis of V did not change the module V but changed the matrix representation of T by similarity (i.e., changed the isomorphism between $\mathrm{GL}(V)$ and $\mathrm{GL}_n(F)$). We introduce the analogous terminology to describe when two representations are the same up to a change of basis.

Definition. *Let (V, φ), (V', φ') be two representations of a group G. A* ***homomorphism*** *or an* ***intertwining operator*** *of two representations is a (continuous) linear transformation $\mathcal{A} : V \to V'$ such that*

$$\mathcal{A}(\varphi(g)v) = \varphi'(g)(\mathcal{A}v) \quad \textit{for any } g \in G,\ v \in V$$

(here V and V' must be vector spaces over the same field F), i.e., the following diagram commutes for any $g \in G$:

$$\begin{array}{ccc} V & \xrightarrow{\varphi(g)} & V \\ {\scriptstyle\mathcal{A}}\downarrow & & \downarrow{\scriptstyle\mathcal{A}} \\ V' & \xrightarrow[\varphi'(g)]{} & V' \end{array}$$

The set of intertwining operators forms a vector space $\mathrm{Hom}(\varphi, \varphi') = \mathrm{Hom}_G(V, V')$, which is a subspace of the space $\mathrm{Hom}_F(V, V')$ of all continuous linear mappings from V to V'.

Two representations of G are ***isomorphic****,* ***equivalent****, or* ***similar****, if there is a linear isomorphism $\mathcal{A}$ which intertwines the two representations. Representations which are not equivalent are called* ***inequivalent****.*

In the above definition, if we identify V and V' as vector spaces, then two representations φ and φ' of G on a vector space V are equivalent if and only if there is some $\mathcal{A} \in \mathrm{GL}(V)$ such that $\mathcal{A} \circ \varphi(g) \circ \mathcal{A}^{-1} = \varphi'(g)$ for all $g \in G$. This $\mathcal{A}$ is a **simultaneous** change of basis for all $\varphi(g)$, $g \in G$. In matrix terminology, two representations φ and φ' are equivalent if there is a fixed invertible matrix P such that

$$P\varphi(g)P^{-1} = \varphi'(g) \quad \text{for all } g \in G.$$

The linear transformation $\mathcal{A}$ or the matrix P above is said to **intertwine** the representations φ and φ' (it gives the "rule" for changing φ into φ').

Exercise. *Let (V, φ) be a representations of a group G. For a fixed $g \in G$ define another representation on the same space V by $\varphi^g(h) \coloneqq \varphi(ghg^{-1})$. Show that φ^g is equivalent to φ.*

1.2 Representations of finite groups

1.2.1 Group algebras

Let F be a field and $G = \{g_1 = e, g_2, \dots, g_n\}$ be a finite group. Recall that, each element of the **group ring** FG is of the form

$$\sum_{i=1}^{n} \alpha_i g_i, \quad \alpha_i \in F.$$

Two formal sums are equal if and only if all corresponding coefficients of group elements are equal. Addition and multiplication in FG are defined as follows:

$$\sum_{i=1}^{n} \alpha_i g_i + \sum_{i=1}^{n} \beta_i g_i := \sum_{i=1}^{n} (\alpha_i + \beta_i) g_i, \quad \Big(\sum_{i=1}^{n} \alpha_i g_i\Big)\Big(\sum_{i=1}^{n} \beta_i g_i\Big) := \sum_{k=1}^{n} \Big(\sum_{\substack{i,j \\ g_i g_j = g_k}} \alpha_i \beta_j \Big) g_k$$

where addition and multiplication of the coefficients α_i and β_j is performed in F.

The group G appears in FG (identifying g_i with $1g_i$) and the field F appears in FG (identifying β with βg_1, where $g_1 = e \in G$). Under these identifications

FG is a vector space over F with the elements of G as a basis.

In particular, $\dim_F FG = |G|$. The elements of F commute with all elements of FG, i.e., F is in the center of FG. When we wish to emphasize the latter two properties we shall say that FG is an F-algebra, called the **group algebra** of G over F (recall that an F-algebra is a ring R which contains F in its center, so R is both a ring and an F-vector space).

Example. *If $G = \langle g \rangle$ is cyclic of order $n \in \mathbb{Z}_{>0}$, then the elements of FG are of the form*

$$\sum_{i=0}^{n-1} \alpha_i g^i.$$

The map $F[x] \to F\langle g \rangle$ which sends x^k to g^k for all $k \geq 0$ extends by F-linearity to a surjective ring homomorphism with kernel equal to the ideal generated by $x^n - 1$. Thus

$$F\langle g \rangle \cong F[x]/(x^n - 1).$$

This is an isomorphism of F-algebras, i.e., is a ring isomorphism which is F-linear.

Let $r = 1 + g + g^2 + \cdots + g^{n-1}$, so $0 \neq r \in F\langle g \rangle$. Note that $rg = g + g^2 + \cdots + g^{n-1} + 1 = r$, hence $r(1 - g) = 0$. Thus the ring $F\langle g \rangle$ contains zero divisors (provided $n > 1$).

More generally, if G is any finite group of order > 1, then for any nonidentity element $g \in G$, $F\langle g \rangle$ is a subring of FG, so FG also contains zero divisors.

Note that the operations in FG are similar to those in the F-algebra $F[x]$ (although $F[x]$ is infinite dimensional over F). In some works FG is denoted by $F[G]$, although the latter notation is currently less prevalent.

Exercise ([1] §18.1 page 842-843)**.** *Show that, if G is a finite group,*

- *the FG-modules V are precisely the representations (V, φ) of G;*
- *the FG-submodules of V are precisely the G-stable subspaces of V;*
- *isomorphic FG-modules are precisely equivalent representations of G.*

Under this correspondence we shall say that the module V ***affords*** *the representation φ of G.*

1.2.2 Complete reducibility

In order to study the decomposition of an FG-module into (direct sums of) submodules we shall need some terminology. We state these definitions for arbitrary rings.

Definition. *Let R be a ring and let M be a nonzero R-module.*

*(1) The module M is said to be **irreducible** (or **simple**) if its only submodules are 0 and M; otherwise M is called **reducible**.*

*(2) The module M is said to be **indecomposable** if M cannot be written as $M_1 \oplus M_2$ for any nonzero submodules M_1 and M_2; otherwise M is called **decomposable**.*

*(3) The module M is said to be **completely reducible** or **semisimple** if it is a direct sum of irreducible submodules. If M is a completely reducible R-module, any direct summand of M is called a **constituent** of M (i.e., N is a constituent of M if there is a submodule N' of M such that $M = N \oplus N'$).*

*A representation is called **irreducible**, **reducible**, **indecomposable**, **decomposable** or **completely reducible** according to whether the FG-module affording it has the corresponding property.*

An irreducible module is, by definition, both indecomposable and completely reducible.

We have seen that, *reducible representations are those with a corresponding matrix representation whose matrices are in block upper triangular form.* Assume further that the representation V of G is decomposable, $V = U \oplus U'$. Take for a basis of V the union of a basis of U and a basis of U'. With this choice of basis the matrix for each $g \in G$ is of the form

$$\varphi(g) = \begin{bmatrix} \varphi_1(g) & 0 \\ 0 & \varphi_2(g) \end{bmatrix}.$$

Thus *decomposable representations are those with a corresponding matrix representation whose matrices are in block diagonal form.*

Example. *For $n > 1$ the FS_n-module $V = F^n$ described earlier (where $\sigma \in S_n$ acts by permuting the subscripts of the basis elements) is reducible since*

$$N := F(e_1 + e_2 + \cdots + e_n) \quad \textit{and} \quad I := \{\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_n e_n \mid \lambda_1 + \lambda_2 + \cdots + \lambda_n = 0\}$$

are proper, nonzero submodules. Moreover V is decomposable since $V = N \oplus I$.

Example. *Let $N := \{\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right] \mid x \in \mathbb{Z}\}$ be the subgroup of strictly upper triangular matrices in $\mathrm{GL}_2(\mathbb{R})$. (It is isomorphic to the additive group $\mathbb{Z}$.) Then the standard representation given by the embedding map $N \hookrightarrow \mathrm{GL}_2(\mathbb{R})$ is of degree 2, acting on $V = \mathbb{R}^2$ with basis e_1, e_2. It is reducible since $\mathbb{R}e_1$ is a proper nonzero submodule, but it is indecomposable. (The minimal polynomial of $\left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right]$ for $x \neq 0$ does not have distinct roots, so the Jordan canonical form of it can only be $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. Since every linear transformation has a unique Jordan canonical form, it cannot be represented by a diagonal matrix, i.e., V is not completely reducible.)*

Similarly, let p be a prime, let $F = \mathbb{F}_p$ and let $G = \langle g \rangle$ be of order p. Let V be the 2-dimensional space over $\mathbb{F}_p$ with basis e_1, e_2 and define an action of g on V by

$$g \cdot e_1 := e_1 \quad \textit{and} \quad g \cdot e_2 := e_1 + e_2.$$

This endomorphism of V does have order p (in $\mathrm{GL}(V)$) and the matrix of g with respect to this basis is the elementary Jordan block $\varphi(g) = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. Now V is reducible (Fe_1 is a G-invariant subspace) but V is indecomposable (the above 2×2 elementary Jordan matrix is not similar to a diagonal matrix).

The first fundamental result in the representation theory of finite groups is the complete reducibility.

Theorem ([1] §18.1 Theorem 1 & Corollaries 2, 3, Maschke's Theorem)**.** *Let G be a finite group and let F be a field whose characteristic does not divide $|G|$. (This assumption applies to any finite group when F has characteristic 0.)*

If V is any FG-module and U is any submodule of V, then V has a submodule W such that $V = U \oplus W$ (i.e., every submodule is a direct summand). Equivalently, every indecomposable representation is irreducible, and every F-representation of G of finite degree is completely reducible.

Let $\varphi : G \to \mathrm{GL}(V)$ be a representation of G of finite degree. Then there is a basis of V such that for each $g \in G$ the matrix of $\varphi(g)$ with respect to this basis is block diagonal:

$$\begin{bmatrix} \varphi_1(g) & & & \\ & \varphi_2(g) & & \\ & & \ddots & \\ & & & \varphi_m(g) \end{bmatrix}$$

where φ_i is an irreducible matrix representation of G, $1 \le i \le m$.

Next we decompose $V = FG$ itself as an FG-module. Then V affords a representation of G of degree equal to $|G|$. If we take the elements of G as a basis of V, then each $g \in G$ permutes these basis elements under the left regular permutation representation:

$$g \cdot g_i \coloneqq g g_i.$$

With respect to this basis of V the matrix of the group element g has a 1 in row i and column j if $g g_j = g_i$, and has 0's in all other positions. This representation is called the **regular representation** of G. Note that each nonidentity element of G induces a nonidentity permutation on the basis of V so the regular representation is always faithful.

To decompose $V = FG$, first of all, we need the characteristic of F not to divide $|G|$. In fact, it will be convenient to have the characteristic of F equal to 0. Secondly, it will simplify matters if we force all the division rings over F to equal the field F (see Schur's Lemma in the next section). Actually the condition that F be algebraically closed is sufficient to ensure this. To simplify notation we shall therefore take $F = \mathbb{C}$ for most of the remainder of the text. The reader can easily check that any algebraically closed field of characteristic 0 (e.g., the field of all algebraic numbers) can be used throughout in place of $\mathbb{C}$.

Theorem ([1] §18.2 Theorems 10 & 12)**.** *Let G be a finite group. Let r be the number of conjugacy classes in G. Then*

(1) G has exactly r inequivalent irreducible complex representations $\varphi_1, \varphi_2, \ldots, \varphi_r$ and they are all finite dimensional.

(2) $\mathbb{C}G$ has exactly r distinct isomorphism types of irreducible submodules $V_{\varphi_1}, V_{\varphi_2}, \ldots, V_{\varphi_r}$ which respectively afford the representation φ_i. More precisely $\mathbb{C}G$ has a decomposition into the direct sum of simple modules:

$$\mathbb{C}G \cong \bigoplus_{i=1}^{r} V_{\varphi_i}^* \otimes V_{\varphi_i} \cong \bigoplus_{i=1}^{r} (V_{\varphi_i})^{\oplus n_i},$$

where n_i is the complex dimension of V_{φ_i} (i.e. the degree of φ_i) for each i. That is to say, the regular representation (over $\mathbb{C}$) of G decomposes as the direct sum of all irreducible representations of G, each appearing with multiplicity equal to the degree of that irreducible representation.

(3) $\sum_{i=1}^{r} n_i^2 = |G|$, *and each* n_i *divides* $|G|$ *for* $i = 1, 2, \ldots, r$.

The collection of all inequivalent irreducible representations will be denoted by $\widehat{G}$. The following corollary shows that $|G| = |\widehat{G}|$ for any finite abelian group G.

Corollary ([1] §18.2 Corollary 11)**.** *Let* G *be a finite abelian group. Then every irreducible complex representation of* G *is* 1*-dimensional (i.e., is a homomorphism from* G *into* $\mathbb{C}^\times$*), and* G *has* $|G|$ *inequivalent irreducible complex representations. Furthermore, every finite dimensional complex matrix representation of* G *is equivalent to a representation into a group of diagonal matrices.*

Proof. The number of conjugacy classes of an abelian group G is $r = |G|$, so $\sum_{i=1}^{r} n_i^2 = |G|$ only if each $n_i = 1$. □

1.3 Schur's Lemma

Any homomorphism of G into the multiplicative group $F^\times = \mathrm{GL}_1(F)$ is a degree 1 (irreducible) matrix representation, called a **character** of G. For example, suppose $G = \langle g \rangle \cong \mathbb{Z}/n\mathbb{Z}$ is the cyclic group of order n and ζ is a fixed n^{th} root of 1 in F. Let $g^i \mapsto \zeta^i$ for all $i \in \mathbb{Z}$. This representation of $\langle g \rangle$ is a faithful representation if and only if ζ is a primitive n^{th} root of 1.

Conversely, Schur's Lemma shows that, any irreducible *complex* representation of a (*finite or infinite*) *abelian* group G is a character of G.

Lemma ([1] §18.2 Lemma 7, Schur's Lemma)**.** *Let* R *be an arbitrary nonzero ring.*

(1) If M *and* N *are simple* R*-modules and* $\varphi : M \to N$ *is a nonzero* R*-module homomorphism, then* φ *is an isomorphism.*

(2) If M *is a simple* R*-module, then* $\mathrm{Hom}_R(M, M)$ *is a division ring.*

Proof. To prove (1) note that since φ is nonzero, $\ker\varphi$ is a proper submodule of M. By simplicity of M we have $\ker\varphi = 0$. Similarly, the image of φ is a nonzero submodule of the simple module N, hence $\varphi(M) = N$. This proves φ is bijective, so (1) holds.

By part (1), every nonzero element of the ring $\mathrm{Hom}_R(M, M)$ is an isomorphism, hence has an inverse. This gives (2). □

Exercise ([1] §18.1 Exercises 17 & 18, Schur's Lemma for complex representations of abelian groups)**.** *Let* G *be an abelian group (*G *may be finite or infinite).*

(1) Show that any irreducible complex representation φ *of* G *is of degree* 1*. Moreover,* $G/\ker\varphi$ *is cyclic if* G *is finite.* [Hint: This can be done by showing that the only finite dimensional division ring over $\mathbb{C}$ is $\mathbb{C}$ itself, or by using the observation that for any $g \in G$ the eigenspaces of $\varphi(g)$ are G-stable. Recall that two diagonalizable matrices are simultaneously diagonalizable if and only they commute.]

(2) If $\varphi : G \to \mathrm{GL}_n(\mathbb{C})$ *is an irreducible matrix representation and* A *is an* $n \times n$ *matrix commuting with* $\varphi(g)$ *for all* $g \in G$*, show that* A *is a scalar matrix.*

(3) Deduce that if φ *is an irreducible complex representation then* $\varphi(z)$ *is a scalar matrix for all elements* $z \in Z(G)$ *in the center of* G*. The character* $\varphi|_{Z(G)} : Z(G) \to \mathbb{C}^\times$ *defined by this scalar is called the* ***central character*** *of* φ*.*

Exercise ([1] §18.1 Exercise 15)**.** *Exhibit all* 1*-dimensional complex representations of a finite cyclic group; make sure to decide which are inequivalent.*

2 Character theory

In this section we show how to attach numerical invariants to linear representations. These invariants depend only on the equivalence class (isomorphism type) of the representation. In other words, for each representation $\varphi : G \to \mathrm{GL}_n(F)$ we shall attach an element of F to each matrix $\varphi(g)$ and we shall see that this number can, in many instances, be computed without knowing the matrix $\varphi(g)$. Moreover, we shall see that these invariants are independent of the similarity class of φ (i.e., are the same for a fixed $g \in G$ if φ is replaced by an equivalent representation) and that they, in some sense, characterize the similarity classes of representations of G.

Throughout this section G is a finite group and, for the moment, F is an arbitrary field. All representations considered are assumed to be finite dimensional.

2.1 Characters

If φ is a representation of G afforded by the FG-module V, the **character** of φ is the function

$$\chi : G \to F \quad \text{defined by} \quad \chi(g) := \mathrm{Tr}\, \varphi(g)$$

where $\mathrm{Tr}\, \varphi(g)$ is the trace of the matrix of $\varphi(g)$ with respect to some basis of V (i.e., the sum of the diagonal entries of that matrix). We shall also refer to χ as the character afforded by the FG-module V. The character is called **irreducible** or **reducible** according to whether the representation is irreducible or reducible, respectively. The **degree** of a character is the degree of any representation affording it.

Example. *The character of the trivial representation is the function $\chi(g) = 1$ for all $g \in G$. This character is called the **principal character** of G.*

*For degree 1 representations, the character and the representation are usually identified (by identifying a 1×1 matrix with its entry). Thus for finite abelian groups, irreducible complex representations and their characters are the **same** (cf. [1] §18.2 Corollary 11).*

*In general, a character of a representation is **not** a character of the finite group: it is **not** a homomorphism from a group into either the additive or multiplicative group of the field!*

Example. *Let $\pi : G \to S_n$ be a permutation representation and let φ be the resulting linear representation on the basis $e_1, \dots, e_n$ of the vector space V:*

$$\varphi(g)(e_i) := e_{\pi(g)i}.$$

With respect to this basis the matrix of $\varphi(g)$ has a 1 in the diagonal entry (i, i) if $\pi(g)$ fixes i; otherwise, the matrix of $\varphi(g)$ has a zero in position (i, i). Thus if χ is the character of φ then

$$\chi(g) = \text{ the number of fixed points of } \pi(g) \text{ on } \{1, 2, \dots, n\}.$$

*In particular, if π is the permutation representation obtained from left multiplication on the set of left cosets of some subgroup H of G then the resulting character is called the **permutation character** of G on H.*

A special case when π is the regular permutation representation of G is worth recording: if φ is the regular representation of G (afforded by the module FG) and ρ is its character, then

$$\rho(g) = \begin{cases} 0 & \text{if } g \neq e, \\ |G| & \text{if } g = e. \end{cases}$$

*The character of the regular representation of G is called the **regular character** of G. Note that this provides specific examples where a character takes on the value 0 and is not a group homomorphism from G into either F or $F^\times$.*

Exercise. *Let G be a finite group, (V,φ) and (V',φ') be representations of G with character χ, χ' respectively.*

(1) ([1] §18.3 Eq.(18.7)) *The direct sum $(V\oplus V', \varphi\oplus\varphi')$ defined by*

$$\varphi\oplus\varphi'(g)(v,v') := (\varphi(g)v, \varphi'(g)v') \quad \textit{for any } g\in G,\ v\in V,\ v'\in V'$$

is also a representation of G. Show that the character of the direct sum $V\oplus V'$ is $\chi+\chi'$. By induction, the character of a representation is the sum of the characters of the constituents appearing in a direct sum decomposition.

(2) ([1] §18.3 Proposition 17) *The tensor product $(V\otimes_F V', \varphi\otimes\varphi')$ defined by*

$$\varphi\otimes\varphi'(g)(v\otimes v') := \varphi(g)v\otimes\varphi'(g)v' \quad \textit{for any } g\in G,\ v\in V,\ v'\in V'$$

is also a representation of G. Show that the character of the tensor product $V\otimes_F V'$ is $\chi\chi'$.

(3) ([3] §2.1 Proposition 3) *For $n\in\mathbb{Z}_{>0}$, the n^{th} tensor power $V^{\otimes n}$ is a representation of G, and the spaces of symmetric tensors $\mathrm{Sym}^n(V)$ and the alternating tensors $\Lambda^n(V)$ are subrepresentations of it. Show that, when $\operatorname{char} F\neq 2$, the characters of $\mathrm{Sym}^2(V)$ and $\Lambda^2(V)$ are*

$$\chi_{\mathrm{Sym}^2(V)}(g) = \frac{\chi(g)^2+\chi(g^2)}{2}, \quad \chi_{\Lambda^2(V)}(g) = \frac{\chi(g)^2-\chi(g^2)}{2}.$$

(4) ([3] §2.1 Exercise 2.3) *The dual space $V^* = \mathrm{Hom}_F(V,F)$ is also a representation of G, called the **contragredient** or **conjugate** of φ, defined by*

$$\big(\varphi^*(g)v^*\big)(v) := v^*\big(\varphi(g^{-1})v\big) \quad \textit{for any } g\in G,\ v\in V,\ v^*\in V^*.$$

Show that, under the dual basis of V^*, the matrix representation is given by $\varphi^*(g) = (\varphi(g)^{-1})^T$, and the character of V^* is $\overline{\chi}$, where the bar denotes complex conjugation.

For $n\times n$ matrices A and B, direct computation shows that $\operatorname{Tr} AB = \operatorname{Tr} BA$. If A is invertible, this implies that $\operatorname{Tr} A^{-1}BA = \operatorname{Tr} B$. Thus the character of a representation is independent of the choice of basis of the vector space affording it, i.e.,

equivalent representations have the same character.

In fact,

Theorem ([1] §18.3 Eq.(18.9))**.** *The correspondence between characters and equivalence classes of complex representations is bijective, i.e., two complex representations are equivalent if and only if they have the same character.*

Let φ be a representation of G of degree n with character χ. Since $\varphi(g^{-1}xg) = \varphi(g)^{-1}\varphi(x)\varphi(g)$ for all $g,x\in G$, taking traces shows that

the character of a representation is a class function.

(A **class function** is any function from G into F which is constant on the conjugacy classes of G, i.e., $f: G\to F$ such that $f(g^{-1}xg) = f(x)$ for all $g,x\in G$.) Moreover,

Theorem ([1] §18.3 Eq.(18.10))**.** *The characters of inequivalent irreducible representations are linearly independent, and they form a basis for the space of all complex class functions.*

Since the trace of the $n \times n$ identity matrix is n and φ takes the identity of G to the identity linear transformation (or matrix),

$$\chi(1) \text{ is the degree of } \varphi.$$

Proposition ([1] §18.3 Proposition 14)**.** *If χ is the character of any complex representation φ of a finite group G, then $\chi(x)$ is a sum of roots of 1 in $\mathbb{C}$ and $\chi(x^{-1}) = \overline{\chi(x)}$ for all $x \in G$ (where the bar denotes complex conjugation).*

Proof. Fix an element $x \in G$ and let $|G| = n$. Since the minimal polynomial of $\varphi(x)$ divides $x^n - 1$ (hence has distinct roots), there is a basis of the underlying vector space such that the matrix of $\varphi(x)$ with respect to this basis is a diagonal matrix with n^{th} roots of 1 on the diagonal. Since $\chi(x)$ is the sum of the diagonal entries (and does not depend on the choice of basis), $\chi(x)$ is a sum of roots of 1. Moreover, if ϵ is a root of 1, $\epsilon^{-1} = \overline{\epsilon}$. Thus the inverse of a diagonal matrix with roots of 1 on the diagonal is the diagonal matrix with the complex conjugates of those roots of 1 on the diagonal. Since the complex conjugate of a sum is the sum of the complex conjugates, $\chi(x^{-1}) = \operatorname{Tr}\varphi(x^{-1}) = \overline{\operatorname{Tr}\varphi(x)} = \overline{\chi(x)}$. □

Keep in mind that in the above proof we first fixed a group element x and then chose a basis of the representation space so that $\varphi(x)$ was a diagonal matrix. It is always possible to diagonalize a single element, but it is possible to simultaneously diagonalize all $\varphi(x)$'s if and only if φ is similar to a sum of degree 1 representations.

2.2 Orthogonality relations for group characters

Let $F = \mathbb{C}$. The next step in the theory of characters is to put an Hermitian inner product structure on the space of class functions and prove that the irreducible characters form an orthonormal basis with respect to this inner product (we already know that they are a basis, cf. [1] §18.3 Eq.(18.10)). For class functions θ and θ' define

$$(\theta, \theta') := \frac{1}{|G|} \sum_{g \in G} \theta(g)\overline{\theta'(g)}.$$

One easily checks that $(\_,\_)$ is Hermitian: for $\alpha_1, \alpha_2 \in \mathbb{C}$,

(a) $(\alpha_1\theta_1 + \alpha_2\theta_2, \theta) = \alpha_1(\theta_1, \theta) + \alpha_2(\theta_2, \theta)$,

(b) $(\theta, \alpha_1\theta_1 + \alpha_2\theta_2) = \overline{\alpha_1}(\theta, \theta_1) + \overline{\alpha_2}(\theta, \theta_2)$, and

(c) $(\theta, \theta') = \overline{(\theta', \theta)}$.

With respect to $(\_,\_)$ the space of class functions is an inner product space, with dimension equal to the number r of conjugacy classes in G.

Theorem ([1] §18.3 Theorem 15, The First Orthogonality Relation for Irreducible Characters)**.** *Let G be a finite group, $\varphi_1, \ldots, \varphi_r$ be the irreducible characters of G over $\mathbb{C}$, and let $\chi_1, \ldots, \chi_r$ be their characters respectively. Then with respect to the inner product $(\_,\_)$ above we have*

$$(\chi_i, \chi_j) = \frac{1}{|G|} \sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and the irreducible characters are an orthonormal basis for the space of class functions. In particular, if θ is any class function then

$$\theta = \sum_{i=1}^{r} (\theta, \chi_i)\, \chi_i.$$

Theorem ([1] §18.3 Theorem 16, The Second Orthogonality Relation for Irreducible Characters)**.** *Under the notation above, for any $x, y \in G$*

$$\sum_{i=1}^{r} \chi_i(x)\overline{\chi_i(y)} = \begin{cases} |C_G(x)| & \text{if } x \text{ and } y \text{ are conjugate in } G \\ 0 & \text{otherwise.} \end{cases}$$

For any class function θ on G, the **norm** of θ is $(\theta, \theta)^{1/2}$ and will be denoted by $\|\theta\|$. When a class function is written in terms of the irreducible characters, $\theta = \sum \alpha_i \chi_i$, its norm is easily calculated as $\|\theta\| = (\sum \alpha_i^2)^{1/2}$. It follows that

a character has norm 1 if and only if it is irreducible.

Finally, observe that computations of the inner product of class functions θ and θ' may be simplified as follows. If $\mathcal{K}_1, \dots, \mathcal{K}_r$ are the conjugacy classes of G with sizes $d_1, \dots, d_r$ and representatives $g_1, \dots, g_r$ respectively, then the value $\theta(g_i)\overline{\theta'(g_i)}$ appears d_i times in the sum for (θ, θ'), once for each element of $\mathcal{K}_i$. Collecting these terms gives

$$(\theta, \theta') = \frac{1}{|G|} \sum_{i=1}^{r} d_i\, \theta(g_i)\overline{\theta'(g_i)},$$

a sum only over representatives of the conjugacy classes of G. In particular,

$$\|\theta\|^2 = (\theta, \theta) = \frac{1}{|G|} \sum_{i=1}^{r} d_i\, |\theta(g_i)|^2.$$

Example. *Let $G = S_3$ and let π be the permutation character of degree 3. Recall that $\pi(\sigma)$ equals the number of elements in $\{1, 2, 3\}$ fixed by σ. The conjugacy classes of S_3 are represented by (1), (12) and (123) of sizes 1, 3 and 2 respectively, and $\pi((1)) = 3$, $\pi((12)) = 1$, $\pi((123)) = 0$. Hence*

$$\|\pi\|^2 = \frac{1}{6}\Big(1 \cdot \pi((1))^2 + 3 \cdot \pi((12))^2 + 2 \cdot \pi((123))^2\Big) = 2.$$

This implies that π is a sum of two distinct irreducible characters, each appearing with multiplicity 1. Let χ_0 be the principal character of S_3, so that $\chi_0(\sigma) = 1$ for all $\sigma \in S_3$. Then

$$(\pi, \chi_0) = \frac{1}{6}\Big(1 \cdot \pi((1))\overline{\chi_0((1))} + 3 \cdot \pi((12))\overline{\chi_0((12))} + 2 \cdot \pi((123))\overline{\chi_0((123))}\Big) = 1$$

so the principal character appears as a constituent of π with multiplicity 1. This proves $\pi = \chi_0 + \chi_2$ for some irreducible character χ_2 of degree 2 (and agrees with our earlier decomposition of this representation). This also shows that $\chi_2(\sigma) = \pi(\sigma) - 1$ for any $\sigma \in S_3$.

*To find the third irreducible character, notice that $\mathrm{sgn} : S_3 \to \{\pm 1\}$ is a representation of degree 1, called the **alternating representation**. We thus get the character table of $G = S_3$:*

classes:	(1)	(12)	(123)
sizes:	1	3	2
χ_0	1	1	1
χ_{sgn}	1	-1	1
χ_2	2	0	-1

The **character table** of a finite group is the table of character values formatted as follows: list representatives of the r conjugacy classes along the top row and list the irreducible characters down the first column. The entry in the table in row χ_i and column g_j is $\chi_i(g_j)$. The character table of a finite group is unique up to a permutation of its rows and columns. (It is customary to make the principal character the first row and the identity the first column and to list the characters in increasing order by degrees.) Also we shall list the size of the conjugacy classes under each class. This will enable one to easily check the "orthogonality of rows" using the first orthogonality relation: if the classes are represented by $g_1, \ldots, g_r$ of sizes $d_1, \ldots, d_r$ then

$$(\chi_i, \chi_j) = \frac{1}{|G|} \sum_{k=1}^{r} d_k \, \chi_i(g_k)\overline{\chi_j(g_k)} = \delta_{ij}.$$

The second orthogonality relation says that the Hermitian product of any two distinct columns of a character table is zero, i.e., it gives an "orthogonality of columns".

Example. *The conjugacy classes of* $G = S_4$ *are represented by* (1), (12), (123), (1234) *and* $(12)(34)$ *of sizes* 1, 6, 8, 6 *and* 3 *respectively. The character table of* S_4 *is the following.*

classes:	(1)	(12)	(123)	(1234)	$(12)(34)$
sizes:	1	6	8	6	3
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	2	0	-1	0	2
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	1	-1

As in the case of S_3*, we have (as with any symmetric group) two representations of degree* 1*: the trivial representation and the alternating representation. The first two characters in the table are the principal character and the sign, respectively.*

Let π *be the natural permutation character of degree* 4 *(so again* $\pi(\sigma)$ *is the number of fixed points of* σ*). Again we compute:*

$$\begin{aligned} \|\pi\|^2 &= \frac{1}{24}\Big(1 \cdot \pi((1))^2 + 6 \cdot \pi((12))^2 + 8 \cdot \pi((123))^2 + 6 \cdot \pi((1234))^2 + 3 \cdot \pi((12)(34))^2\Big) \\ &= \frac{1}{24}(1 \cdot 4^2 + 6 \cdot 2^2 + 8 \cdot 1^2 + 6 \cdot 0^2 + 3 \cdot 0^2) = 2 \end{aligned}$$

so π *has two distinct irreducible constituents. If* χ_1 *is the principal character of* S_4*, then*

$$(\pi, \chi_1) = \frac{1}{24}\Big(1 \cdot \pi((1)) + 6 \cdot \pi((12)) + 8 \cdot \pi((123)) + 6 \cdot \pi((1234)) + 3 \cdot \pi((12)(34))\Big) = 1.$$

This proves that the degree 4 *permutation character is the sum of the principal character and an irreducible character (listed above as* χ_4*) of degree* 3*.*

Notice that $\chi_5 = \chi_4\chi_2$*. It is irreducible because* $\|\chi_5\| = 1$*. At last* χ_3 *can be completed with* $\sum_{i=1}^{r}(\deg \chi_i)^2 = |G|$*:*

$$24 = 1^2 + 1^2 + \chi_3((1))^2 + 3^2 + 3^2 \quad \Longrightarrow \quad \chi_3((1)) = 2,$$

and with the second orthogonality relations, for example,

$$0 = \sum_{i=1}^{r} \chi_i((1))\overline{\chi_i((12))} = 1 \cdot 1 + 1 \cdot (-1) + 2 \cdot \overline{\chi_3((12))} + 3 \cdot 1 + 3 \cdot (-1) \implies \chi_3((12)) = 0.$$

Exercise ([1] §19.1)**.** *Calculate the character table of* $\mathbb{Z}/4\mathbb{Z}$*,* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$*, and* A_4*.*

2.3 Fourier expansions on finite abelian groups

We make some observations which show how representation theory for finite groups corresponds to "Fourier series" for some infinite groups (in particular, to Fourier series on the circle).

Let G be the multiplicative group of points on the unit circle in $\mathbb{C}$:

$$G = S^1 := \{z \in \mathbb{C} \mid |z| = 1\}.$$

We shall usually view G as the interval $[0, 2\pi]$ in $\mathbb{R}$ with the two end points identified, i.e., as the additive group $\mathbb{R}/2\pi\mathbb{Z}$ (the isomorphism is: the real number x corresponds to the complex number e^{ix}). Note that G has a translation invariant measure, namely the Lebesgue measure, and the measure of the circle is 2π. For finite groups, the counting measure is the translation invariant measure (so the measure of a subset H is $|H|$, the number of elements in that subset) and integrals on a finite group with respect to this counting measure are just finite sums.

The space

$$L^2(G) := \{f : G \to \mathbb{C} \mid f \text{ is measurable and } |f|^2 \text{ is integrable over } G\}$$

plays the role of the group algebra of the infinite group G. This space becomes a commutative ring with 1 under the convolution of functions: for any function $f, g \in L^2(G)$ the product $f*g : G \to \mathbb{C}$ is defined by

$$(f * g)(x) := \frac{1}{2\pi} \int_0^{2\pi} f(x-y)g(y)\,dy \quad \text{for all } x \in G.$$

Recall that for a finite group H, the group algebra $\mathbb{C}H$ is also formally the ring of $\mathbb{C}$-valued functions on H under a convolution multiplication and that these functions are written as formal sums — the element $\sum \alpha_g g \in \mathbb{C}H$ denotes the function which sends g to $\alpha_g \in \mathbb{C}$ for all $g \in H$.

Recall that for an abelian group, all irreducible representations are 1-dimensional; and for 1-dimensional representations, characters and representations may be identified. The complete set of continuous homomorphisms of G into $\mathrm{GL}_1(\mathbb{C})$ is given by

$$e_n(x) := e^{inx}, \quad x \in [0, 2\pi], \quad n \in \mathbb{Z}.$$

The ring $L^2(G)$ admits an Hermitian inner product: for $f, g \in L^2(G)$

$$(f, g) := \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}\,dt.$$

Under this inner product, $\{e_n \mid n \in \mathbb{Z}\}$ is an orthonormal basis (of the Hilbert space $L^2(G)$, where the term "basis" is used in the analytic sense that these are independent and 0 is the only function orthogonal to all of them). Moreover,

$$L^2(G) = \widehat{\bigoplus_{n \in \mathbb{Z}}} \mathbb{C}e_n$$

where the hat over the direct sum denotes taking the closure of the direct sum in the L^2-topology, and equality indicates equality in the L^2 sense. (Recall that, similarly, the group algebra of a finite abelian group is the direct sum of the irreducible 1-dimensional submodules, each occurring with multiplicity one). These facts imply the well-known result from Fourier analysis, that every square-integrable function $f(x)$ on $[0, 2\pi]$ has a Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

where the Fourier coefficients c_n are given by

$$c_n := (f, e_n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int}\, dt.$$

This brief description indicates how the representation theory of finite groups extends to certain infinite groups and the results we have proved may already be familiar in the latter context. In fact, there is a completely analogous theory for arbitrary (not necessarily abelian) compact Lie groups — here the irreducible (complex) representations need not be 1-dimensional but they are all finite dimensional and $L^2(G)$ decomposes as a direct sum of them, each appearing with multiplicity equal to its degree. The emphasis (at least at the introductory level) in this theory is often on the importance of being able to represent functions as (Fourier) series and then using these series to solve other problems (e.g., solve differential equations). The underlying group provides the "symmetry" on which to build this "harmonic analysis", rather than being itself the principal object of study.

We end this section with an introduction to Fourier expansions on any finite abelian groups H. Consider the Hilbert space

$$L^2(H) := \{f : H \to \mathbb{C}\} \cong \mathbb{C}H$$

with an Hermitian inner product defined by

$$(f, g) := \frac{1}{|H|} \sum_{t \in H} f(t)\overline{g(t)} \quad \text{for any } f, g \in L^2(H).$$

Note that any complex-valued function on H is measurable and square-integrable since H is finite and the measure is the (normalized) counting measure.

Again, a character χ on a group H is a group homomorphism $\chi : H \to \mathbb{C}^\times$ and $\widehat{H}$ is the collection of characters. For f a complex-valued function on H, the **Fourier transform** $\widehat{f}$ of f is the function on $\widehat{H}$ defined by

$$\widehat{f}(\chi) := (f, \chi) = \frac{1}{|H|} \sum_{t \in H} f(t)\chi^{-1}(t) \quad \text{for any } \chi \in \widehat{H}.$$

(By Lagrange's Theorem $g^{|H|} = e$ for any g in the finite group H, and hence $\chi(g)^{|H|} = \chi(g^{|H|}) = \chi(e) = 1$ for any $\chi \in \widehat{H}$, therefore $|\chi(g)| = 1$. In particular χ is **unitary** in the sense that $\chi(g)^{-1} = \overline{\chi(g)}$.) The **Fourier expansion** or **Fourier series** of f is

$$f \sim \sum_{\chi \in \widehat{H}} \widehat{f}(\chi)\chi.$$

Theorem ([2] Theorem 4.0.1)**.** *On a finite abelian group H, the Fourier expansion of a complex-valued function f* ***represents*** *f, in the sense that, for every $g \in H$,*

$$f(g) = \sum_{\chi \in \widehat{H}} \widehat{f}(\chi)\chi(g).$$

The elements of $\widehat{H}$ form an orthogonal basis for $L^2(H)$. In particular, the Fourier coefficients are unique.

Proof. The result is a direct corollary of the orthogonality of characters (cf. [1] §18.3 Theorem 15). In the following we give another proof of the statement that, all the characters $\chi \in \widehat{H}$ form a basis of $L^2(H)$.

The group operation in H will be written multiplicatively, not additively, to fit better with other notational conventions. For our convenient, we consider the right regular representation of H on $\mathbb{C}H$:

$$g \cdot \sum_{h \in H} \alpha_h h \coloneqq \sum_{h \in H} \alpha_h h g^{-1}.$$

Given by the isomorphism $L^2(H) \cong \mathbb{C}H$ (which maps the function $f : H \to \mathbb{C}$ to the formal sum $\sum_{h \in H} f(h)h$), the right regular representation of H can be presented by **right translation** on $L^2(H)$:

$$(g \cdot f)(x) \coloneqq f(xg) \quad \text{for all } f \in L^2(H),\ g, x \in H.$$

By [1] §18.2 Theorems 10, $L^2(H)$ has a decomposition into the direct sum of simple modules. Since all simple modules are of degree 1, they all appear only once in the decomposition of $L^2(H)$.

For any representation V of H and any character $\chi \in \widehat{H}$, let the χ-**eigenspace** in V be

$$V_\chi \coloneqq \{v \in V \mid g \cdot v = \chi(g)v \text{ for any } g \in G\}.$$

It is easy to check that $\chi \in L^2(H)_\chi$; and for any $f \in L^2(H)_\chi$,

$$\begin{aligned} f(x) = f(e \cdot x) &= (x \cdot f)(e) && \text{(by definition of right translation on } L^2(H)) \\ &= \chi(x) f(e) && \text{(by definition of the } \chi\text{-eigenspace } L^2(H)_\chi) \end{aligned}$$

i.e., $f = f(e)\chi$. Therefore the irreducible constituent $L^2(H)_\chi$ of $L^2(H)$ equivalent to χ is simply $\mathbb{C}\chi$, the 1-dimensional subspace of $L^2(H)$ spanned by χ. The decomposition can be written as

$$L^2(H) = \bigoplus_{\chi \in \widehat{H}} L^2(H)_\chi = \bigoplus_{\chi \in \widehat{H}} \mathbb{C}\chi.$$

□

Other related exercises in [1]

§18.1 2 3 4 6 7 8 10
§18.2 13 14
§18.3 2 5 6 10 11 12 13 27
§19.1 1 3 4 5 6 15

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